MATHEMATICAL THEORIES

OF

PLANETARY MOTIONS

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D. E.

PREFACE.

The determination of the motions of the heavenly bodies is an important problem in and for itself, and also on account of the influence it has exerted on the development of mathematics. It has engaged the attention of the greatest mathematicians and in the course of their not altogether successful attempts to solve it, they have displayed unsurpassed ingenuity. The methods devised by them have proved useful not only in this problem but have also largely determined the course of advance in other branches of mathematics. Analytical mechanics, beginning with Newton and receiving a finished clearness from Lagrange, is especially indebted to this problem, and in turn, analytical mechanics has been so suggestive in method as to determine largely both the direction and rapidity of the advancement of mathematical science.

Hence when it is desired to illustrate the abstract theories of analytical mechanics, the profundity of the mathematics of the problem of the motions of the heavenly bodies, its powerful influence on the historical development of this science, and finally the dignity of its object, all point to it as most suitable for this purpose.

This work is intended not merely as an introduction to the special study of astronomy, but rather for the student of mathematics who desires an insight into the creations of his masters in this field. The lack of a text-book, giving within moderate limits and in a strictly scientific manner the principles of mathematical astronomy in their present remarkably simple and lucid form, is undoubtedly the reason why so many mathematicians extend their knowledge of the solar system but little beyond Kepler's laws. The author has endeavored to meet this need and at the same time to produce a book which shall be so near the present state of the science as to include recent investigations and to indicate unsettled questions.

The subject of the work is that part of celestial mechanics which treats of the motions of heavenly bodies considered as material points. This is its most important part, and it is fundamental in the theories of rotation, tides and the figures of bodies. The author hopes to treat of these in a separate work. The simplest processes and those which best represent the present state of the science have always been selected. Especial care has been taken to guard against brilliant hypotheses which explorers in this field have so often indulged in but which are not suitable in a text-book.

References to original sources of information are invariably given. These will be useful to students who desire to study further. Assistance in this direction is also afforded by the historical sketches accompanying each important sub-division of the work.

The tables at the end of the book give the numerical values of the elements of the solar system according to Leverrier and Newcomb.

For the pecuniary aid which permitted the necessary studies and the publication of this work, the author begs to return his sincere thanks to his Excellency, Dr. von Gossler, Minister of Religious, Educational and Medical affairs.

CHARLOTTENBURG, September 1, 1888.

DR. DZIOBEK.

NOTES BY THE TRANSLATORS.

The author kindly consented to read the proof of this translation. Many changes have been introduced by him.

M. W. HARRINGTON.

The publication of this translation has been delayed in a number of ways. The author kindly offered to assist in reading the proofs and at the same time to revise the work. By reason of his remoteness much time was consumed in sending the proofs back and forth. Moreover, the changes which he introduced in the text were so numerous and of such a character that it became necessary to reset the type of large portions of the earlier forms. To such an extent was this the case that, after eighty pages had been printed, it was decided to send a type-written copy of the remainder of the translation to the author and to have him revise it before setting the type. This was done with its accompanying delay. In the meantime Professor Harrington had become Chief of the Weather Bureau, and at this juncture I was asked to revise the remainder of the translation, to incorporate in it the author's corrections and additions and to see it through the press.

W. J. HUSSEY.

PALO ALTO, CAL., July 8, 1892.

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ERRATA.

PAGE	LINE	FOR	READ
10	—6, —7	*motus*	*Motus.*
11	+8	†	*
13	headline	PROBLEMS	PROBLEM
17	"	"	"
21	+5	$\int_{\phi_1}^{\phi}$	$\int_{\phi_1}^{\phi_0}$
21	+9	$\left((x-a+\sqrt{x(-2a)}\right)$	$\left(\frac{(x-a)\pm\sqrt{x(x-2a)}}{-a}\right)$
22	+11	f_1	f
22	—4	t_0	t_0
24	+10	smallest	smallest value
27	headline	PROBLEMS	PROBLEM
31	"	"	"
31	+6	$\left(\frac{xz}{3}\right)^3$	$\left(\frac{xz}{2}\right)^3$
40	+19	m_1	m_λ
42	+15	$0=$	$c=$
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49	+9	qualities	quantities
80	—2	$=0.$	.
115	—4	(15)	15.
176	—1, —2	*Sq*	*Sg*
212	+4	W	*W*
228	—8	g_λ	g_a

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MATHEMATICAL
THEORIES OF PLANETARY MOTIONS.

FIRST DIVISION.

Solution of the Problem of Two Bodies. Formation of the General Integrals for the Problem of n Bodies. Algebraic Transformations of this Problem.

1. Newton's Law of Gravitation. Motion of Two Points Subject to It.

Newton's law of gravitation is the point of departure in mathematical investigations of the motions of the heavenly bodies. This law reads as follows:

Each particle of matter attracts any other particle with a force whose magnitude is directly as the product of their masses and inversely as the square of their distance from each other.

Assume that P_1 and P_2 are two gravitating particles, the coordinates and mass of the first, referred to stationary rectangular coordinates, are x_1, y_1, z_1, m_1, and those of the second are x_2, y_2, z_2, m_2, then the distance between them is

$$(1) \qquad r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

and this is to be considered as always positive. The total force between the bodies, according to Newton's law,

$$= k^2 \frac{m_1 m_2}{r^2},$$

where k is a constant whose magnitude depends on the selected units of mass, distance, and time.

Since the two particles attract each other, the direction of action is along the line which joins them. The action of P_2 on P_1 has the direction from P_1 to P_2 and its direction cosines are

$$\frac{x_2 - x_1}{r}, \qquad \frac{y_2 - y_1}{r}, \qquad \frac{z_2 - z_1}{r}.$$

The direction of the action of P_1 on P_2 is exactly the reverse, and its direction cosines are

$$\frac{x_1 - x_2}{r}, \qquad \frac{y_1 - y_2}{r}, \qquad \frac{z_1 - z_2}{r}.$$

The components of the first force in the direction of the coordinate axes are therefore

$$k^2 m_1 m_2 \frac{x_2 - x_1}{r^3}, \quad k^2 m_1 m_2 \frac{y_2 - y_1}{r^3}, \quad k^2 m_1 m_2 \frac{z_2 - z_1}{r^3}.$$

The components of the second force are, also,

$$k^2 m_1 m_2 \frac{x_1 - x_2}{r^3}, \quad k^2 m_1 m_2 \frac{y_1 - y_2}{r^3}, \quad k^2 m_1 m_2 \frac{z_1 - z_2}{r^3}.$$

Consequently the differential equations of the motion of the two points are, when t represents the time,

$$(2) \qquad \left\{ \begin{array}{l} m_1 \dfrac{d^2 x_1}{dt^2} = k^2 m_1 m_2 \dfrac{x_2 - x_1}{r^3}, \\[2ex] m_1 \dfrac{d^2 y_1}{dt^2} = k^2 m_1 m_2 \dfrac{y_2 - y_1}{r^3}, \\[2ex] m_1 \dfrac{d^2 z_1}{dt^2} = k^2 m_1 m_2 \dfrac{z_2 - z_1}{r^3}, \\[2ex] \text{and also} \\[2ex] m_2 \dfrac{d^2 x_2}{dt^2} = k^2 m_1 m_2 \dfrac{x_1 - x_2}{r^3}, \\[2ex] m_2 \dfrac{d^2 y_2}{dt^2} = k^2 m_1 m_2 \dfrac{y_1 - y_2}{r^3}, \\[2ex] m_2 \dfrac{d^2 z_2}{dt^2} = k^2 m_1 m_2 \dfrac{z_1 - z_2}{r^3}. \end{array} \right.$$

These differential equations are valid for any system of coordinates. Hence the letters x, y, z, can be cyclically interchanged. Advantage will be taken of this to simplify the

manner of writing the succeeding equations, in that only one of the three equations will be written, leaving the others to be made from it by exchange of letters. These cases will be indicated by the sign *. Introducing this change, equations (2) become

$$(2) \qquad * \begin{cases} m_1 \dfrac{d^2x_1}{dt^2} = k^2\, m_1\, m_2\, \dfrac{x_2 - x_1}{r^3}, \\ \\ m_2 \dfrac{d^2x_2}{dt^2} = k^2\, m_1\, m_2\, \dfrac{x_1 - x_2}{r^3}. \end{cases}$$

These are six total and simultaneous differential equations and the determination of the motion of the two bodies is reduced to their integration. As they are all of the second order their complete integration will introduce twelve arbitrary constants. That this number is necessary appears directly from the fact that, to make the problem a definite one, twelve conditions must be expressed, for instance, six coordinates and six component velocities, for any given instant.

To prepare for the integration of equations (2), add the first and fourth. This gives

$$* \quad m_1 \frac{d^2 x_1}{dt^2} + m_2 \frac{d^2 x_2}{dt^2} = 0.$$

Integrating this once, and calling the constant $*\alpha_x$, we get

$$(3) \qquad * \quad m_1 \frac{dx_1}{dt} + m_2 \frac{dx_2}{dt} = \alpha_x.$$

These equations, on a second integration, give

$$(4) \qquad * \quad m_1\, x_1 + m_2\, x_2 = \alpha_x t + \beta_x.$$

Equations (4), in which $*\alpha_x$, β_x are constants, have a simple interpretation. If the coordinates of the center of gravity of the two points, that is the point which divides their distance in the inverse ratio of their masses, are X, Y, Z, then

$$(5) \qquad * \quad X = \frac{m_1\, x_1 + m_2\, x_2}{m_1 + m_2},$$

and equations (4) become

$$(6) \qquad * \quad (m_1 + m_2)\, X = \alpha_x t + \beta_x.$$

From these equations it appears that *the center of gravity*

moves in a straight line with uniform velocity,—which is the law of the conservation of the motion of the center of gravity.

Returning to equations (2), if we refer P_2 to a moving system of rectangular coordinates parallel to the original with the origin constantly at P_1, and designate the coordinates of P_2 in the new system by x, y, z, we have

$$(7) \qquad * \quad x = x_2 - x_1,$$

and we observe that these equations are included in the right-hand members of equations (2).

Put therefore

$$(8) \qquad k^2 (m_1 + m_2) = \mu,$$

divide the first three equations of (2) by m_1, the second three by m_2, and subtract in pairs, and we get

$$(9) \qquad * \quad \frac{d^2x}{dt^2} = -\mu \frac{x}{r^3}.$$

These three equations by integration afford the six remaining constants and complete the solution of the problem. For, by solving (5) and (7) for $*x_1$, x_2, we have

$$(10) \qquad * \quad x_1 = X - \frac{m_2}{m_1 + m_2} x, \quad x_2 = X + \frac{m_1}{m_1 + m_2} x.$$

P_1 may be considered as representing the sun, P_2 the planet. Equations (9) then determine the (relative) motion of the planet about the sun, and the motion is as if the sun were fixed and the planet attracted by the sum of the masses of the two bodies.

To integrate (9), multiply the second by $-z$, the third by $+y$, and add. We thus get

$$y \frac{d^2z}{dt^2} - z \frac{d^2y}{dt^2} = 0.$$

This is at once integrable, and gives

$$(11) \qquad * \quad y \frac{dz}{dt} - z \frac{dy}{dt} = c\zeta_x,$$

where $c\zeta_x$ is the constant of integration.

These are the so-called sectorial integrals. $ydz - zdy$ is twice the area of the triangle in the plane of yz, whose angles are the origin, the projection of the point x, y, z, and the projection

of the adjacent point $x + dx$, $y + dy$, $z + dz$. It is positive or negative, according as the infinitesimal angle at the origin is described by the radius vector in the direction from $+ y$ to $+ z$, or the reverse. If this triangle is named dS_x then

$$* \quad 2dS_x = c\zeta_x dt.$$

To fix the coordinate system, we will, once for all, so place it that a rotation from $+ y$ to $+ z$ is contra-clockwise when the clock face is in the plane yz and is directed toward $+ x$: and similarly for rotations from $+ z$ to $+ x$, and from $+ x$ to $+ y$. A sector is then positive when the direction of description is contra-clockwise, and *vice versa*.

Equations (11) show that the areas described by the projections of the radius vector are proportional to the times. For the unit of time these areas are $\frac{c\zeta_x}{2}$, $\frac{c\zeta_y}{2}$, $\frac{c\zeta_z}{2}$. From this it follows that these areas themselves are proportional to the times and that, for the unit of time, they are $\frac{c}{2}$, where c is positive, and, as is always possible,

(12) $$1 = \zeta^2_x + \zeta^2_y + \zeta^2_z.$$

The expression $y\frac{dz}{dt} - z\frac{dy}{dt}$ is called the moment of velocity about the x axis. It is easily seen that it is equal to the product of the projection of the velocity on the yz plane and the perpendicular from the origin on its direction. The same is true for the two other analogous expressions. Hence the square root of the sum of the squares of these three expressions, that is, c, equals the moment of velocity about the origin.

From (11), by multiplication with x, y, z, and addition, we get a final equation between the co-ordinates,

(13) $$\zeta_x x + \zeta_y y + \zeta_z z = 0,$$

and hence

The path of a planet is a plane curve and is so traced that the areas described by the radius vector from the sun are proportional to the times. It follows that ζ_x, ζ_y, ζ_z are the direction cosines of a line which we shall call the ζ-axis and which is normal to the plane of motion.

For further integration it is best to form an equation in r and t only.

$$r^2 = x^2 + y^2 + z^2,$$

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt},$$

and by another differentiation

$$\frac{d\left(r\dfrac{dr}{dt}\right)}{dt} = x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2} + \frac{dx^2 + dy^2 + dz^2}{dt^2},$$

or, by (9), when for brevity

(14) $$r\frac{dr}{dt} = r',$$

(15) $$\frac{dr'}{dt} = -\frac{\mu}{r} + \frac{dx^2 + dy^2 + dz^2}{dt^2}.$$

Differentiating again,

$$\frac{d^2r'}{dt^2} = +\frac{\mu\dfrac{dr}{dt}}{r^2} + 2\frac{dxd^2x + dyd^2y + dzd^2z}{dt^3},$$

or, by (9),

(16) $$\frac{d^2r'}{dt^2} = -\frac{\mu r'}{r^3}.$$

This equation has a significant relation with equations (9), in that x, y, z, are replaced by r'. If it is combined with (9) in the same way that the latter were combined with each other to get the sectorial law, and, when the three constants of integration are represented by $f\xi_x$, $f\xi_y$, $f\xi_z$, we get

(17) $$* \quad x\frac{dr'}{dt} - r'\frac{dx}{dt} = f\xi_x.$$

Here it will be assumed that f is positive and that $\xi_x^2 + \xi_y^2 + \xi_z^2 = 1$, so that ξ_x, ξ_y, ξ_z become the direction cosines of the ξ-axis which lies in the plane of motion and, as we shall soon see, is directed to the perihelion.

If we multiply equations (9) by dx, dy, dz, add and integrate, we get the expression for the kinetic energy

(18) $$\frac{1}{2}\;\frac{dx^2 + dy^2 + dz^2}{dt^2} = \frac{\mu}{r} - \frac{\mu}{2a},$$

where the constant of integration, for reasons which will soon appear, is represented by $-\frac{\mu}{2a}$.

Between the integrals (11), (17) and (18), exist two identical relations. From (11) and (17) we get at once

(19) $$\xi_x \zeta_x + \xi_y \zeta_y + \xi_z \zeta_z = 0.$$

Also,

$$c^2 = c^2\zeta_x^2 + c^2\zeta_y^2 + c^2\zeta_z^2 = (x^2 + y^2 + z^2)\frac{dx^2 + dy^2 + dz^2}{dt^2} - \left(\frac{xdx + ydy + zdz}{dt}\right)^2,$$

or, by (18),

(20) $$c^2 = r^2\left(2\frac{\mu}{r} - \frac{\mu}{a}\right) - r'^2.$$

Also,

(21) $$f^2 = f^2\xi_x^2 + f^2\xi_y^2 + f\xi_z^2,$$

and then, from (17) and (18),

$$f^2 = r^2\left(\frac{dr'}{dt}\right)^2 + r'^2\left(2\frac{\mu}{r} - \frac{\mu}{a}\right) - 2r'^2\frac{dr'}{dt},$$

or, since by (15) and (18)

(22) $$\frac{dr'}{dt} = \frac{\mu}{r} - \frac{\mu}{a},$$

(23) $$f^2 = r^2\left(\frac{\mu}{r} - \frac{\mu}{a}\right)^2 + r'^2\frac{\mu}{a}.$$

From (20) and (23) follows the second relation between the integrals,

(24) $$c^2\frac{\mu}{a} + f^2 = \mu^2.$$

In order finally to get an equation between the coordinates, multiply (17) in order by x, y, z, and add. With the help of (20) and (22), we then get

(25) $$c^2 - \mu r = f(x\xi_x + y\xi_y + z\xi_z).$$

This equation represents a surface of revolution of the second degree which is cut in a conic by the plane (13), which plane by (19) passes through the η-axis of the surface. Since,

also, by (25) it appears that the origin is a focus of the surface, we have

The path of the planet about the sun is a conic and the sun is in one of its foci.

We have now two lines perpendicular to each other, the ξ- and ζ-axis, which are fully determined by the motion, and we shall now take a third line, the η-axis, perpendicular to the others, so that ξ, η, ζ form a new system of rectangular coordinates. Let η_x, η_y, η_z be the three direction cosines which the η-axis makes with the axes of the first system, then the nine quantities

$$\begin{matrix} \xi_x, & \xi_y, & \xi_z, \\ \eta_x, & \eta_y, & \eta_z, \\ \zeta_x, & \zeta_y, & \zeta_z, \end{matrix} \tag{26}$$

are the coefficients of transformation from the old system into the new. This transformation may be written in either form

$$\begin{aligned} * \quad x &= \Xi\,\xi_x + H\,\eta_x + Z\,\zeta_x, \\ * \quad \Xi &= x\,\xi_x + y\,\xi_y + z\,\zeta_z. \end{aligned} \tag{26a}$$

To find the expressions for η_x, η_y, η_z we may use the known relations between the nine quantities (26); for instance,

$$* \quad \eta_x = \zeta_y\,\xi_z - \zeta_z\,\xi_x,$$

or, with (11) and (17) and a slight transformation,

$$* \quad c\,f\,\eta_x = x\,\frac{d\rho}{dt} - \rho\,\frac{dx}{dt} \tag{17a}$$

where

$$\rho = -\,r^2\,\frac{dr'}{dt} + r'^2 = -\,c^2 + \mu r. \tag{14a}$$

From equations (17a) a very interesting result can be easily deduced. Writing them

$$* \quad c\,\frac{dx}{dt} - f\,\eta_x = -\,\frac{\mu}{c}\left(x\,\frac{dr}{dt} - r\,\frac{dx}{dt}\right) = -\,\mu\,\frac{y\,\zeta_z - z\,\zeta_y}{r},$$

we get

$$\left(\frac{dx}{dt} - f\,\frac{\eta_x}{c}\right)^2 + \left(\frac{dy}{dt} - f\,\frac{\eta_y}{c}\right)^2 + \left(\frac{dz}{dt} - f\,\frac{\eta_z}{c}\right)^2 = \left(\frac{\mu}{c}\right)^2. \tag{25a}$$

We have also

$$\frac{dx}{dt}\zeta_x + \frac{dy}{dt}\zeta_y + \frac{dz}{dt}\zeta_z = 0, \tag{13a}$$

and it appears at once that

The hodograph of the motion of a planet is a circle with the radius $\frac{\mu}{c}$, *and its plane is the plane of motion.*

The equations of the orbit are, in the new system,

$$\zeta = 0,$$

$$c^2 - \mu\, r = f\,\xi. \tag{27}$$

Also,

$$\xi \frac{d\eta}{dt} - \eta \frac{d\xi}{dt} = c. \tag{28}$$

Introducing polar coordinates by putting

$$\begin{cases} \xi = r \cos v, \\ \eta = r \sin v, \end{cases} \tag{29}$$

we get (27) and (28) in the following forms

$$r = \frac{\frac{c^2}{\mu}}{1 + \frac{f}{\mu}\cos v}, \tag{30}$$

$$\frac{r^2\, dv}{dt} = c. \tag{31}$$

If r and v are found by integration of (30) and (31) we have ξ and μ by (29), and x, y, z by (26a). As $\zeta = 0$, we evidently have

$$* \quad x = \Xi\, \xi_x + H\, \eta_x = r\,(\cos v\, \xi_x + \sin v\, \eta_x). \tag{26b}$$

Assuming, with Gauss,

$$\begin{cases} \xi_x = A_x \cos B_x, \\ \eta_x = A_x \sin B_x \end{cases} \tag{32}$$

and we get

$$* \quad x = r\, A_x \cos\,(v + B_x). \tag{32a}$$

The six constants A and B satisfy the three equations

$$A_x^2 + A_y^2 + A_z^2 = 2,$$
$$A_x^2 \sin 2\,B_x + A_y^2 \sin 2\,B_y + A_z^2 \sin 2\,B_z = 0,$$
$$A_x^2 \cos 2\,B_x + A_y^2 \cos 2\,B_y + A_z^2 \cos 2\,B_z = 0,$$

because (32a) must give identically $r^2 = x^2 + y^2 + z^2$.

If p is the semi-parameter and e the eccentricity of a conic, its equation, when the origin is at the focus and the + axis is directed to the vertex nearest it, becomes

$$r = \frac{p}{1 + e \cos v}. \tag{30a}$$

Hence, in this case,

$$p = \frac{c^2}{\mu}, \tag{33}$$

$$e = \frac{f}{\mu}. \tag{34}$$

Inserting in (33) for μ its value (8), we get

$$k = \frac{c}{\sqrt{p} \sqrt{m_1 + m_2}}.$$

Since c is double the sector described in a unit of time, if S equals the sector described in the time T, then $c = \frac{2S}{T}$, and hence

$$k = \frac{2S}{T\sqrt{p} \sqrt{m_1 + m_2}},$$

and

$$k^2 = \frac{\left(\frac{2S}{T}\right)^2}{p\,(m_1 + m_2)}. \tag{35}$$

Hence *

For the different planets moving about the sun, the square of the ratio of the sectorial velocity to the time is as the product of the semi-parameter into the sum of the masses of sun and planet.

The constant k is *Gauss's constant.* If the sun's mass is unity, that of the earth† is $\frac{1}{354710} = 0.0000028192$. The unit of distance is half the major axis of the earth's orbit, which is also

* See Gauss's *Werke*, (*Theoria motus*, etc.), Vol. VII, page 12.

† The numerical constants in this translation are from Gauss's *Theoria motus*. While values more acceptable to astronomers, at present, might be inserted yet, as the object of the book is rather a correct analytical than numerical development of the problems involved, and astronomers are not generally agreed as to the exact numerical values employed, I have preferred to leave these numbers generally unchanged.—TRANSLATOR.

the *mean distance* of the earth from the sun. The sector becomes in a sidereal year the entire area of the ellipse, or

$$S = \pi a b = \pi a \sqrt{a} \sqrt{\frac{b^2}{a}} = \pi a \sqrt{a} \sqrt{p} = \pi \sqrt{p}.$$

If the mean solar day is the unit of time, T is the number of such days in a sidereal year, or

$$T = 365.2563835,$$

and by computation

$$k = 0.017202099.\dagger$$

Gauss also proposed so to select the unit of mass that k should equal unity. This is evidently the case when, retaining the above units of distance and time, the sun's mass is assumed as $= (0.017202099)^2$. The unit of mass can then be defined as the mass which, at distance unity, during an interval of unity, gives an acceleration $= 1$. For if $k = 1$, then, from (2),

$$\sqrt{\left(\frac{d^2x_2}{dt^2}\right)^2 + \left(\frac{d^2y_2}{dt^2}\right)^2 + \left(\frac{d^2z_2}{dt^2}\right)^2} = \frac{m_1}{r^2}.$$

From this it appears that the unit of mass is fixed when the choice of the units of distance and time has been made.

The farther treatment of the problem requires that r and v be expressed in terms of the time t. It follows directly from (30a) and (31), with the aid of (33), that

$$(36) \qquad dt = \frac{p^{\frac{3}{2}}\, dv}{\sqrt{\mu}\,(1 + e \cos v)^2}.$$

If t_0 is the time of passage through the vertex nearest the sun, at which time $v = 0$, the integration of (36) gives

$$(37) \qquad t - t_0 = \frac{p^{\frac{3}{2}}}{\sqrt{\mu}} \int_0^v \frac{dv}{(1 + e \cos v)^2}.$$

This integral requires different treatment when e equals, is greater than, or is less than, unity. The three cases will be considered separately.

* For a more exact value of k, and for its exact signification, see Oppolzer, *Lehrbuch der Bahnbestimmungen*, Vol. I, page 45.

2. Elliptic, Parabolic and Hyperbolic Orbits.

Case I, $e < 1$.

Equation (30a), § 1, represents in this case an ellipse. If the semi-axis major is represented by $[a]$, then

$$[a] = \frac{p}{1 - e^2} = \frac{\frac{c^2}{\mu}}{1 - \frac{f^2}{\mu^2}}. \quad \text{Hence by (24), § 1,}$$

$$[a] = a,$$

that is, the constant a is positive and equals the semi-axis major of the ellipse. Equation (18) therefore shows that *the velocity of a planet depends only on its major axis and its distance from the sun.*

The denominator of (37) then never $= 0$ and v increases continuously, though not uniformly, with t. It can be evaluated directly as a trigonometrical integral, but it is simpler when we introduce an auxiliary angle E, such that

$$\text{(1)} \qquad \tan \tfrac{1}{2} v \sqrt{\frac{1 - e}{1 + e}} = \tan \tfrac{1}{2} E.$$

By differentiation this gives

$$\text{(2)} \qquad \frac{dv}{\cos^2 \frac{1}{2} v} \sqrt{\frac{1 - e}{1 + e}} = \frac{dE}{\cos^2 \frac{1}{2} E} = dE\,(1 + \tan^2 \tfrac{1}{2} E).$$

E increases continuously with v and, if we take $E = 0$ when $v = 0$, the two angles will be equal for multiples of π and will always be in the same semi-circumference.

From (1) it follows that

$$\text{(3)} \qquad \cos^2 \tfrac{1}{2} v = \frac{1}{1 + \tan^2 \frac{1}{2} E \left(\frac{1 + e}{1 - e}\right)},$$

and (2) passes into

$$\text{(4)} \quad dv = \sqrt{1 - e^2}\, \frac{1 + \tan^2 \frac{1}{2} E}{1 - e + \tan^2 \frac{1}{2} E\,(1 + e)}\, dE = \frac{\sqrt{1 - e^2}}{1 - e \cos E}\, dE.$$

Further,

$$\text{(5)} \qquad 1 + e \cos v = 1 - e + 2 e \cos^2 \frac{v}{2} = \frac{1 - e^2}{1 - e \cos E},$$

and also, from (37), § 1,

$$t - t_0 = \frac{p^{\frac{3}{2}}}{\sqrt{\mu}\,(1 - e^2)^{\frac{3}{2}}} \int_0^E dE\,(1 - e\cos E)$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}\,(E - e\sin E).$$

If we make

(6) $$\frac{a^{\frac{3}{2}}}{\sqrt{\mu}} = \frac{1}{n}, \text{ and}$$

(7) $$n\,(t - t_0) = M,$$

we get

(8) $$M = E - e\sin E.$$

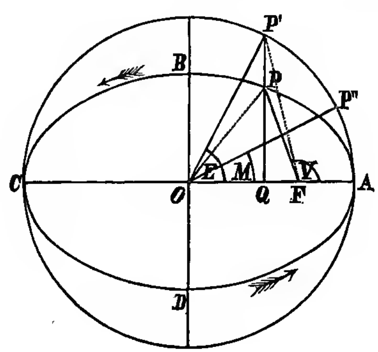
Equation (8) is *Kepler's equation.* It does not directly give what is wanted, that is E as a function of t. To obtain this, the equation must be solved for E, a problem which is discussed in § 4. When this is done v can be got from (1), r from (30a), ξ and η from (29), and finally x, y, z, by transformations of (26b) or (32a).

The three angles v, M and E, are called by astronomers,

(12) $$\begin{cases} v, \text{ the true anomaly,} \\ M, \text{ the mean anomaly,} \\ E, \text{ the eccentric anomaly.} \end{cases}$$

Also, n is the mean daily motion.

The anomalies have a very simple geometrical meaning.



Let $ABCD$ be the ellipse of a planet, which moves in the direction of the arrow, and let F be the focus occupied by the sun. Let the planet be at P, then angle $AFP = v$, the true anomaly. Extend the perpendicular QP above P to P' where it cuts the circle described on the major axis, then angle $QOP' = E$, the eccentric anomaly. For

$$OQ + QF = a\,e = a \cos E - r \cos v = a \cos E - \frac{p \cos v}{1 + e \cos v},$$

which passes into equation (5).

Imagine, finally, on the circle, a point moving with uniform velocity and reaching A and C at the same time as the planet. Let P'' be its position when the planet is at P; then angle $AOP'' = M$, the mean anomaly, and n is the daily increase of the mean anomaly.

To get the periodic time T, the year of the planet, we note that in this time M becomes $2\,\pi$, and hence

$$2\pi = T\,n, \text{ and, by (6),}$$

(13) $$\frac{T^2}{a^3} = \frac{4\pi^2}{\mu}.$$

If, as is the fact, m_2 is so small as compared with m_1 that it can be neglected in the first approximation, the right hand member of (13) is the same for all planets, and we reach the celebrated *Laws of Kepler:*

1. *The orbits of the planets are ellipses, in one focus of each of which is the sun.*

2. *The areas described by the radii vectores from the sun in equal times are equal.*

3. *The squares of the periodic times of the different planets are as the cubes of their mean distances.*

The relations between r, v, M, E, e, a, can be given different forms adapted to special requirements. Putting

(14) $$e = \sin \varphi,$$

the most important of these, given by Gauss in his *Theoria Motus*, are the following:

$$
(15)\begin{cases}
\text{I.} & p = a\cos^2\varphi, \\
\text{II.} & r = \dfrac{p}{1 + e\cos v}, \\
\text{III.} & r = a\,(1 - e\cos E), \\
\text{IV.} & \cos E = \dfrac{\cos v + e}{1 + e\cos v}, \text{ or } \cos v = \dfrac{\cos E - e}{1 - e\cos E}, \\
\text{V.} & \sin\tfrac{1}{2}E = \sqrt{\dfrac{1 - \cos E}{2}} = \sin\tfrac{1}{2}v\sqrt{\dfrac{1 - e}{1 + e\cos v}} \\
& = \sin\tfrac{1}{2}v\sqrt{\dfrac{r\,(1 - e)}{p}} = \sin\tfrac{1}{2}v\sqrt{\dfrac{r}{a\,(1 + e)}}, \\
\text{VI.} & \cos\tfrac{1}{2}E = \sqrt{\dfrac{1 + \cos E}{2}} = \cos\tfrac{1}{2}v\sqrt{\dfrac{1 + e}{1 + e\cos v}} \\
& = \cos\tfrac{1}{2}v\sqrt{\dfrac{r\,(1 + e)}{p}} = \cos\tfrac{1}{2}v\sqrt{\dfrac{r}{a\,(1 - e)}}, \\
\text{VII.} & \tan\tfrac{1}{2}E = \tan\tfrac{1}{2}v\tan(45^\circ - \tfrac{1}{2}\varphi), \\
\text{VIII.} & \sin E = \dfrac{r\sin v\cos\varphi}{p} = \dfrac{r\sin v}{a\cos\varphi}, \\
\text{IX.} & r\cos v = a\,(\cos E - e) \\
& = 2a\cos(\tfrac{1}{2}E + \tfrac{1}{2}\varphi + 45^\circ)\cos(\tfrac{1}{2}E - \tfrac{1}{2}\varphi - 45^\circ), \\
\text{X.} & \sin\tfrac{1}{2}(v - E) = \sin\tfrac{1}{2}\varphi\sin v\sqrt{\dfrac{r}{p}} \\
& = \sin\tfrac{1}{2}\varphi\sin E\sqrt{\dfrac{a}{r}}, \\
\text{XI.} & \sin\tfrac{1}{2}(v + E) = \cos\tfrac{1}{2}\varphi\sin v\sqrt{\dfrac{r}{p}} \\
& = \cos\tfrac{1}{2}\varphi\sin E\sqrt{\dfrac{a}{r}}, \\
\text{XII.} & M = E - e\sin E.
\end{cases}
$$

Finally, we will collect and name the six constants which astronomers call the elements of the orbit. They are

(16)
1. The mean distance or semi-axis major $= a$,
2. The eccentricity $= e$,
3. The longitude of the ascending node $= \Omega$,
4. The inclination of the plane of the orbit $= i$,
5. The longitude of perihelion $= \pi$,
6. The mean longitude of the planet at the selected epoch $= \varepsilon$.

The constants a and e have the same meaning as already given to them. From now on we will so select the xy plane that $c\zeta_x = \dfrac{xdy - ydx}{dt}$ will always be positive. It divides the plane of the orbit into two parts, of which the one on the $+ z$ side is the upper, the other the lower part. The point where the planet passes from the lower to the upper part is the ascending node, and the opposite the descending node. The longitude of the ascending node, Ω, is then the angle between the $+ x$ axis and the radius vector directed to the ascending node. The inclination i, is the acute angle which the plane of the orbit makes with the plane of xy.

The equation of the plane of the orbit then becomes

$$x \sin \Omega \sin i - y \cos \Omega \sin i + z \cos i = 0.$$

When $i = 0$, Ω is indeterminate. If ω is the angle which the perihelion makes with the ascending node, measured on the plane of the orbit, then the longitude of perihelion is measured in part, (Ω), on the xy plane, in part, (ω), on the plane of the orbit. Hence

$$\pi = \Omega + \omega. \tag{17}$$

This designation appears singular but is convenient, especially when i is very small. When $i = 0$, π is the angle between the $+ x$ axis and the perihelion.

If we take ω instead of π, the nine coefficients of transformation become

$$\tag{18} \begin{cases} \xi_x = \cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i, \\ \xi_y = \cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i, \\ \xi_z = \sin \omega \sin i, \\ \eta_x = - \sin \omega \cos \Omega - \cos \omega . \sin \Omega \cos i, \\ \eta_y = - \sin \omega \sin \Omega + \cos \omega \cos \Omega \cos i, \\ \eta_z = \cos \omega \sin i, \\ \zeta_x = \sin \Omega \sin i, \\ \zeta_y = - \cos \Omega \sin i, \\ \zeta_z = \cos i. \end{cases}$$

The mean anomaly and time have the relation

$$M = nt - nt_0.$$

For t_0 let us take a new constant ε such that

(20) $$-nt_0 = \varepsilon - \pi, \text{ and}$$

(21) $$M = nt + \varepsilon - \pi.$$

To $nt + \varepsilon$ is given the name of *mean longitude*, (measured from the $+x$ axis), and we may make

(22) $$\zeta = nt + \varepsilon.$$

All the formulas are simplified when $e = 0$, that is when the orbit is a circle, and the motion is uniform. If, at the same time, the plane of xy is the plane of the orbit, both Ω and π become arbitrary, and $nt + \varepsilon = \zeta$ becomes the angle made by the planet with the $+x$ axis.

Introducing polar coordinates into (26b), we have

(23) $$\begin{cases} x = r\,[\cos(v + \pi) + \sin(v + \pi - \Omega)\sin\Omega\,(1 - \cos i)], \\ y = r\,[\sin(v + \pi) - \sin(v + \pi - \Omega)\cos\Omega\,(1 - \cos i)], \\ z = r\sin(v + \pi - \Omega)\sin i, \end{cases}$$

in which form they are convenient for computation.

Case II, $e = 1$.

The orbit is a parabola; a is infinite, and M and $E = 0$, or lose their significance. Equation (37), § 1, becomes easily integrable by substituting

$$\tan\tfrac{1}{2}v = z.$$

The integral then becomes

$$\int_0^v \frac{dv}{(1 + \cos v)^2} = \tfrac{1}{2}\int_0^z dz\,(1 + z^2) = \tfrac{1}{2}\left(z + \frac{z^3}{3}\right),$$

and hence

(24) $$\frac{2\,(t - t_0)\sqrt{\mu}}{p^{\frac{3}{2}}} = \tan\tfrac{1}{2}v + \tfrac{1}{3}\tan^3\tfrac{1}{2}v.$$

This equation must be solved for v and the farther solution of the problem then proceeds as in the ellipse. For $t = -\infty$, $v = -\pi$ and $r = +\infty$; for $t = +\infty$, $v = +\pi$, and $r = +\infty$; hence it appears that when $\dfrac{1}{a} = 0$, according to (18), § 1, the

planet comes from infinity with infinitesimal velocity, reaches its greatest velocity at perihelion, then withdraws again to infinity with a velocity which gradually decreases to 0.

Case III, $e > 1$.

Equation (30a), § 1, represents a hyperbola. Since r is positive, we must have $1 + e \cos v > 0$, hence $\cos v > -\frac{1}{e}$. Putting $\cos \psi = \frac{1}{e}$, v must vary from $-180° + \psi$ to $180° - \psi$. At these limits $r = \infty$. If v is taken beyond $180° - \psi$, equation (30a), § 1, gives negative values of r, and we pass to the branch of the hyperbola which does not inclose the sun. This plays no part in the physical problem with which we are engaged, since the planet remains continuously on the first branch.

If we call $[a]$ the semi-axis major of the hyperbola, it follows that

$$[a] = \frac{p}{e^2 - 1},$$

or, by (33), (34) and (24), § 1,

$$[a] = -a,$$

that is, the constant a is negative in the hyperbola, and its absolute value is that of the semi-axis major. M and E become imaginary; that is, they are illusory in the physical problem. Yet an auxiliary angle F can be introduced through the equation

$$(25) \qquad \tan \tfrac{1}{2} F = \sqrt{\frac{e-1}{e+1}} \tan \tfrac{1}{2} v = \tan \tfrac{1}{2} \psi \tan \tfrac{1}{2} v,$$

and from (37), § 1, we get

$$(26) \qquad \frac{(t - t_0)\sqrt{\mu}}{(-a)^{\frac{3}{2}}} = e \tan F - \log_e [\tan (\tfrac{1}{2} F + 45°)].$$

Hence, using the expression

$$(27) \qquad u = \tan (\tfrac{1}{2} F + 45°),$$

$$(28) \qquad \frac{(t - t_0)\sqrt{\mu}}{(-a)^{\frac{3}{2}}} = \frac{e}{2}\left(u - \frac{1}{u}\right) - \log_e u.$$

When u has been determined by (28), F and v can be got from (27) and (25). From (18), § 1, it appears that the planet comes from infinity with the velocity $\sqrt{\frac{\mu}{-a}}$, accelerates its velocity to perihelion, and then returns to infinity with a velocity which continuously decreases until it becomes $\sqrt{\frac{\mu}{-a}}$.

Case I represents the orbits of all the planets moving about the sun. The two others occur with comets and meteorites which move sometimes in ellipses which are usually very long, sometimes in parabolas and sometimes in narrow hyperbolas.

3. The Rectilinear Path and the Formula of Lambert and Euler.

In addition to the cases of the preceding paragraphs, there may be another, that in which $c\zeta_x = c\zeta_y = c\zeta_z = 0$. In this case equations (11), § 1, can be at once integrated, and we get, if p_1, p_2, p_3 are the three constants,

$$\frac{x}{p_1} = \frac{y}{p_2} = \frac{z}{p_3}, \tag{1}$$

or the equations of a straight line passing through the sun. If we take this for the x axis, y and $z = 0$, and

$$r = \pm x.$$

If the planet is on the positive side of the origin, it must pass through the sun to reach the negative side. At the instant it reaches the sun, however, the force becomes infinite, the principles of the Calculus lose their significance, and the discussion closes at this point. We can then always put

$$r = + x.$$

The differential equation of the motion becomes

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}. \tag{2}$$

This can be at once integrated, on multiplying by dx, and gives

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{\mu}{x} - \frac{\mu}{2a}, \tag{3}$$

where $-\frac{\mu}{2a}$ is again the constant of integration.

We will now so select our time that when $x = x_0$, $t = t_0$ and $\frac{dx}{dt}$, the velocity, is negative. The planet is then approaching the sun and (3) becomes

$$dt = -\frac{dx}{\sqrt{2\mu}\sqrt{\frac{1}{x}-\frac{1}{2a}}}.$$

Designating, finally, by t_1 the time at which the planet reaches x_1, and reversing the limits of integration, we have

$$t_1 - t_0 = \frac{1}{\sqrt{2\mu}}\int_{x_1}^{x_0}\frac{dx}{\sqrt{\frac{1}{x}-\frac{1}{2a}}}. \tag{3a}$$

1. If a is positive, then must $x \leqq 2a$. Making, therefore,

$$x = 2a \sin^2 \tfrac{1}{2}\varphi, \quad \text{or} \quad \sqrt{x(2a-x)} = a \sin\varphi, \tag{3b}$$

we get

$$t_1 - t_0 = \frac{2a^{\frac{3}{2}}}{\sqrt{\mu}}\int_{\phi_1}^{\phi_0} d\varphi \sin^2\tfrac{1}{2}\varphi = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}(\varphi - \sin\varphi)\Big|_{\phi_1}^{\phi_0}$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}\left[\text{arc sin}\,\frac{\sqrt{x(2a-x)}}{a} - \frac{\sqrt{x(2a-x)}}{a}\right]_{x_1}^{x_0}.$$

Putting $x_0 = 2a$ and $x_1 = 0$, or letting the body fall freely to the sun from a distance $2a$, the time necessary for the fall becomes, since $\varphi_1 = 0$ and $\varphi_0 = \pi$,

$$T = \frac{a^{\frac{3}{2}}\pi}{\sqrt{\mu}}, \tag{4}$$

or, by (13), § 2, T is half the periodic time of a planet moving in an ellipse with the mean distance a.

2. If $a = \infty$, we get at once

(5) $$t_1 - t_0 = \frac{2}{3\sqrt{2\mu}}\left(x_0^{\frac{3}{2}} - x_1^{\frac{3}{2}}\right).$$

3. If a is negative, we may put

$$x = -2a\left(\frac{e^{\frac{\phi}{2}} - e^{-\frac{\phi}{2}}}{2}\right)^2,$$

and then

$$t_1 - t_0 = \frac{2(-a)^{\frac{3}{2}}}{\sqrt{\mu}}\int_{\phi_1}^{\phi}\left(\frac{e^{\frac{\phi}{2}} - e^{-\frac{\phi}{2}}}{2}\right)^2 d\varphi$$

$$= \frac{(-a)^{\frac{3}{2}}}{\sqrt{\mu}}\left(\frac{e^{\phi} - e^{-\phi}}{2} - \varphi\right)\Bigg|_{\phi_1}^{\phi_0}$$

or

(6) $$t_1 - t_0 = \frac{[-a]^{\frac{3}{2}}}{\sqrt{\mu}}\left[\frac{\sqrt{x(x-2a)}}{-a} - \log_e\left((x-a) + \sqrt{x(x-2a)}\right)\right]_{x_1}^{x_0}.$$

If the initial velocity is directed away from the sun, there are also three cases:

1. If a is positive. The velocity $= 0$, when $x = 2a$, and the planet turns back toward the sun at this point.

2. If $a = \infty$. The velocity $= 0$ when r becomes ∞; that is, the planet goes off to infinity with a velocity which gradually decreases to 0.

3. If a is negative. The velocity continues positive and decreases to $\sqrt{\frac{\mu}{-a}}$, and with this velocity it disappears in infinity.

From these considerations it appears that even when the path is rectilinear, it can be considered as an infinitely narrow ellipse, parabola, or hyperbola, according as $\frac{1}{a} \gtreqless 0$.

It is now interesting to note that the formula, (3a), is correct for the general motion in any orbit, if x_0 and x_1 have in the general case other but simple values, becoming in the rectilinear path simple distances from the sun. We shall prove it for the ellipse. It is clear that the time of motion between two points depends only on the distances of these points from the sun, their distances apart, and the semi-axis a. To find its expression in terms of these quantities, represent what depends on one point by the subscript 0, and on the other by the subscript 1. Then, if we put

$$\frac{E_1 - E_0}{2} = f_1 \text{ and } \frac{E_1 + E_0}{2} = g.$$

We find, by simple transformations,

$$M_1 - M_0 = \sqrt{\frac{\mu}{a^3}}\,(t_1 - t_0) = 2\,(f - e\cos g\sin f),$$
$$r_1 + r_0 = 2a\,(1 - e\cos g\cos f),$$
$$\rho = 2a\,(\sqrt{1 - e^2\cos^2 g})\sin f,$$
$$r_1 - r_0 = 2a\,e\sin g\sin f.$$

To express $t_1 - t_0$ by r_0, r_1, ρ, and a, we must eliminate e, f, and g from the above equations, and as in the first three e and g are contained only in the form $e\cos g$, we have the result:

The time of motion between two points depends only on a, $r_1 + r_0$, and ρ.

If we put $e\cos g = \cos \alpha$ and then

$$\alpha + f = \varphi_0, \quad \alpha - f = \varphi_1$$

we find immediately

$$t_1 - t_0 = \frac{2a^{\frac{3}{2}}}{\sqrt{\mu}}\,(\varphi - \sin\varphi)\Big|_{\phi_1}^{\phi_0},$$

and

$$\frac{r_1 + r_0 + \rho}{2a} = \sin^2\frac{\varphi_0}{2},$$

$$\frac{r_1 + r_0 - \rho}{2a} = \sin^2\frac{\varphi_1}{2}.$$

Comparing now (3a) with those which follow, we find, by reversing the process of integration,

$$
(7) \qquad t_1 - t_0 = \int_{\frac{r_1 + r_0 - \rho}{2}}^{\frac{r_1 + r_0 + \rho}{2}} \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{2a}}},
$$

which is Lambert's theorem. We shall find it in § 18 by quite another process.

4. Solution of Kepler's Equation. Development of the Coordinates as Functions of the Time.

In order to develop the coordinates as functions of the time, it is necessary, first of all, to solve Kepler's equation,

$$
(1) \qquad M = E - e \sin E,
$$

with reference to E. Numerous methods of solving this celebrated equation have been devised, when M and e are given, and the simplest of these is that given by Gauss in his *Theoria Motus*. The later applications require a more general solution, and there are two roads that lead to this, the one through Lagrange's series, (which seems to have been discovered by Lagrange in his study of Kepler's equation), the other through Bessel's functions.

Lagrange's series enables us to develop x in ascending powers of e, when the two have the relation expressed by the equation

$$
(2) \qquad x = y + e\,\varphi(x),
$$

where $\varphi(x)$ is any given function of x. The development of this equation runs as follows,—

$$
(3) \qquad x = y + \frac{e}{1}\,\varphi(y) + \frac{e^2}{2\,!}\,\frac{\partial[\varphi(y)]^2}{\partial y} + \frac{e^3}{3\,!}\,\frac{\partial^2[\varphi(y)]^3}{\partial y^2} + \frac{e^4}{4\,!}\,\frac{\partial^3[\varphi(y)]^4}{\partial y^3} + \ldots .
$$

This series can be generalized by developing in ascending powers of e any function, $f(x)$, in the place of x. It then becomes

$$(4)\quad f(x) = f(y) + \frac{e}{1}\cdot\varphi(y)\,f'(y) + \frac{e^2}{2\,!}\,\frac{\partial[\{\varphi(y)\}^2 f'(y)]}{\partial y} + \frac{e^3}{3\,!}\,\frac{\partial^2[\{\varphi(y)\}^3 f'(y)]}{\partial y^2} + \ldots.$$

The question of the convergence of this series is one into which we shall not enter, but it has occupied the attention of mathematicians. The final result (given, for instance, by Serret in his *Algèbre supérieure*, Fifth Edition, Vol. I, page 480) is as follows:

The equation $x = y + e\,\varphi(x)$, when e and y are given, has several roots, and these are generally different. If e_1 is the smallest for which, with a given value of y, two of these roots are equal, then Lagrange's series is convergent, so long as the absolute value (modulus) of e is smaller than that of e_1. In that case indeed, the series gives the root which has the smallest modulus.

In the case of Kepler's problem, we must put

$$x = E, \quad y = M, \quad \varphi(x) = \sin x,$$

and the series becomes

$$(5)\qquad E = M + \frac{e}{1}\sin M + \frac{e^2}{2\,!}\,\frac{\partial \sin^2 M}{\partial M} + \frac{e^3}{3\,!}\,\frac{\partial^2 \sin^3 M}{\partial M^2} + \frac{e^4}{4\,!}\,\frac{\partial^3 \sin^4 M}{\partial M^3}.$$

To carry out the differentiations, it is convenient to develop the powers of $\sin M$ in terms of the sines and cosines of multiples of M. We then get

$$\begin{aligned}(6)\qquad E = M &+ e\sin M + \frac{e^2}{2!\,2}\,2\sin 2M \\ &+ \frac{e^3}{3!\,2^2}\,(3^2\sin 3M - 3\sin M) \\ &+ \frac{e^4}{4!\,2^3}\,(4^3\sin 4M - 4\times 2^3\sin 2M) \\ &+ \frac{e^5}{5!\,2^4}\,(5^4\sin 5M - 5\times 3^4\sin 3M + 10\sin M) \\ &+ \frac{e^6}{6!\,2^5}\,(6^5\sin 6M - 6\times 4^5\sin 4M + 15\times 2^5\sin 2M),\end{aligned}$$

and this is exact up to the sixth power of e.

To get r, the formula $r = a\,(1 - e\cos E)$ may be used. In this case, in equation (4), $f(x) = \cos x$, and we get

$$(7)\quad \frac{r}{a} = 1 - e\cos M + \frac{e^2}{2}(1 - \cos 2M)$$
$$- \frac{e^3}{2!\,2^2}(3\cos 3M - 3\cos M)$$
$$- \frac{e^4}{3!\,2^3}(4^2\cos 4M - 4\times 2^2\cos 2M)$$
$$- \frac{e^5}{4!\,2^4}(5^3\cos 5M - 5\times 3^3\cos 3M + 10\cos M)$$
$$- \frac{e^6}{5!\,2^5}(6^4\cos 6M - 6\times 4^4\cos 4M + 15\times 2^4\cos 2M).$$

From this, since $\xi = \dfrac{a\,(1-e^2) - r}{e}$, we get at once

$$(8)\quad \frac{\xi}{a} = \cos M - \frac{e}{2}(3 - \cos 2M) + \frac{e^2}{2!\,2^2}(3\cos 3M - 3\cos M)$$
$$+ \frac{e^3}{3!\,2^3}(4^2\cos 4M - 4\times 2^2\cos 2M)$$
$$+ \frac{e^4}{4!\,2^4}(5^3\cos 5M - 5\times 3^3\cos 3M + 10\cos M)$$
$$+ \frac{e^5}{5!\,2^5}(6^4\cos 6M - 6\times 4^4\cos 4M + 15\times 2^4\cos 2M)$$
$$+ \frac{e^6}{6!\,2^6}(7^5\cos 7M - 7\times 5^5\cos 5M$$
$$+ 21\times 3^5\cos 3M - 35\cos M).$$

Also from $\eta = a\sqrt{1-e^2}\sin E = a\sqrt{1-e^2}\,\dfrac{E-M}{e}$ follows

$$(9)\quad \frac{\eta}{a\sqrt{1-e^2}} = \sin M + \frac{e}{2!\,2}\,2\sin 2M$$
$$+ \frac{e^2}{3!\,2^2}(3^2\sin 3M - 3\sin M)$$
$$+ \frac{e^3}{4!\,2^3}(4^3\sin 4M - 4\times 2^3\sin 2M)$$
$$+ \frac{e^4}{5!\,2^4}(5^4\sin 5M - 5\times 3^4\sin 3M + 10\sin M)$$
$$+ \frac{e^5}{6!\,2^5}(6^5\sin 6M - 6\times 4^5\sin 4M + 15\times 2^5\sin 2M)$$
$$+ \frac{e^6}{7!\,2^6}(7^6\sin 7M - 7\times 5^6\sin 5M + 21\times 3^6\sin 3M - 35\sin M).$$

In order to get η itself in ascending powers of e, series (9) must be multiplied by the expansion of $\sqrt{1-e^2}$:

$$(1-e^2)^{\frac{1}{2}} = 1 - \frac{e^2}{2} - \frac{e^4}{8} - \frac{e^6}{16},$$

and this gives

$$\begin{aligned}
(10) \qquad \frac{\eta}{a} &= \sin M + \frac{e}{2}\sin 2M \\
&+ \frac{e^2}{8}(3\sin 3M - 5\sin M) \\
&+ \frac{e^3}{12}(4\sin 4M - 5\sin 2M) \\
&+ \frac{e^4}{384}(125\sin 5M - 153\sin 3M - 22\sin M) \\
&+ \frac{e^5}{240}(81\sin 6M - 104\sin 4M + 10\sin M) \\
&+ \frac{e^6}{6!\,2^6}(7^5\sin 7M - 23125\sin 5M \\
&\qquad + 4887\sin 3M + 2285\sin M).
\end{aligned}$$

Somewhat more difficult is the formation of the expression for v, and the development of the so-called *equation of the center*, by which name astronomers designate the difference between the true and mean anomalies.

We have the relation $\tan\frac{1}{2}v\sqrt{\dfrac{1-e}{1+e}} = \tan\frac{1}{2}E$. By the introduction of exponentials, and representing 2.71828 . . . by $[e]$, this equation takes the form

$$\frac{[e]^{iv}-1}{[e]^{iv}+1} = \sqrt{\frac{1+e}{1-e}}\,\frac{[e]^{iE}-1}{[e]^{iE}+1},$$

hence, making $\dfrac{\sqrt{1+e}-\sqrt{1-e}}{\sqrt{1+e}+\sqrt{1-e}} = \dfrac{e}{1+\sqrt{1-e^2}} = k,$

$$[e]^{iv} = [e]^{iE}\,\frac{1-k[e]^{-iE}}{1-k[e]^{iE}}.$$

Taking the logarithms, this becomes

$$iv = iE + \log_e (1 - k[e]^{-iE}) - \log_e (1 - k[e]^{iE}).$$

Since $k < 1$ and the modulus of $[e]^{iE} = 1$, these logarithms can be developed in series, and, passing finally to the trigonometrical functions, we get

$$v = E + 2\left(k \sin E + \frac{k^2}{2} \sin 2E + \frac{k^3}{3} \sin 3E + \ldots\right).$$

To get v in ascending powers of e, the beginning must be made with k.

$$k = \frac{e}{1 + \sqrt{1 - e^2}}: \text{ that is, } k = \frac{e}{2} + \frac{e}{2} k^2.$$

Insert in Lagrange's formula $x = k$, $y = \frac{e}{2}$, $\varphi(x) = \frac{k^2}{2}$, and, if $f(x) = x^i$, we get

$$k^i = \frac{e^i}{2^i} + \frac{ie^{i+2}}{2^{i+2}} + \frac{i(i+3)}{2!}\frac{e^{i+4}}{2^{i+4}} + \frac{i(i+4)(i+5)}{3!}\frac{e^{i+6}}{2^{i+6}} + \ldots$$

If now E, $\sin E$, $\sin 2E \ldots$ are developed in ascending powers of e and the proper reductions made, the expression for v becomes

$$\begin{aligned}
(11) \quad v = {} & M + e \times 2 \sin M + e^2 \times \frac{5}{4} \sin 2M \\
& + e^3 \left(\frac{13}{12} \sin 3M - \frac{1}{4} \sin M\right) \\
& + e^4 \left(\frac{103}{96} \sin 4M - \frac{11}{24} \sin 2M\right) \\
& + e^5 \left(\frac{1097}{960} \sin 5M - \frac{43}{64} \sin 3M + \frac{5}{96} \sin M\right) \\
& + e^6 \left(\frac{1223}{960} \sin 6M - \frac{451}{480} \sin 4M + \frac{17}{192} \sin 2M\right).
\end{aligned}$$

Before the series (6) to (11) are used, their convergence must be assured. For this purpose the criterion mentioned on page 24 will serve. The equation

$$E = M + e \sin E$$

has two equal roots if the two differential quotients of both sides are equal.

$$1 = e \cos E, \quad \cos E = \frac{1}{e}, \quad \sin E = \sqrt{1 - \frac{1}{e^2}} = \frac{i}{e}\sqrt{1-e^2},$$

hence

$$\text{arc} \cos \frac{1}{e} = M + i\sqrt{1-e^2}, \text{ or,}$$

$$(12) \quad \frac{1}{e} = \cos(M + i\sqrt{1-e^2}) = \cos M \cos(i\sqrt{1-e^2})$$

$$- \sin M \sin(i\sqrt{1-e^2}) = \cos M \frac{[e]^{-\sqrt{1-e^2}} + [e]^{+\sqrt{1-e^2}}}{2}$$

$$- \sin M \frac{[e]^{-\sqrt{1-e^2}} - [e]^{+\sqrt{1-e^2}}}{2i}.$$

If $M = 0$, the equation becomes

$$\frac{1}{e} = \frac{[e]^{-\sqrt{1-e^2}} + [e]^{+\sqrt{1-e^2}}}{2},$$

the smallest root of which is $e = 1$.

If $M = \frac{\pi}{2}$,

$$\frac{1}{e} = \frac{[e]^{+\sqrt{1-e^2}} - [e]^{-\sqrt{1-e^2}}}{2i},$$

or, if we put $e = i\alpha$,

$$2 = \alpha\left([e]^{+\sqrt{1+\alpha^2}} - [e]^{-\sqrt{1+\alpha^2}}\right),$$

and from this, solving for $[e]^{\sqrt{1+\alpha^2}}$,

$$(13) \qquad 1 + \sqrt{1+\alpha^2} = \alpha\,[e]^{\sqrt{1+\alpha^2}}.$$

The smallest root of this equation is

$$\alpha = 0.66195\ldots, \quad e = 0.66195\,i.$$

If any other value of M is taken, a root for e is obtained from (12), whose modulus is between 0.66195 and 1. Hence the conclusion:

If $e < 0.66195$, the series (6) to (11) converge in every case.

If $1 > e > 0.66195$, these series converge only for a part of the path.

If $e > 1$, the series diverge in all cases.

For the major planets, moving about the sun, e is always a

very small fraction, and these series are serviceable, but for many of the asteroids their convergence is tedious. In the latter case another method is in use among astronomers for overcoming the difficulties of Kepler's equation.

It is possible, for each value of e between 0 and 1, as shown by Bessel,* to form a progressive series in terms of the sines and cosines of the multiples of M, the coefficients of which are progressive series in terms of the ascending powers of e. Such series can evidently be made from series (6) to (11), of which the law of formation is evident. These series can be looked on as doubly infinite series, and all the terms can be collected together which contain the angle $n\,M$. In this way, in (6) for instance, the coefficient of $\sin M$

$$= e - \frac{e^3}{1!\,2!\,2^2} + \frac{e^5}{2!\,3!\,2^4} - \frac{e^7}{3!\,4!\,2^6} + \ldots$$

To make the discussion general, and especially to show the convergence of the series, it is convenient to begin with Fourier's theorem for the development of a function in a trigonometrical series.

Since $E - M = e \sin E$ remains unchanged when M is increased by 2π, the value of $E - M$, according to this theorem, must, for all values of E, be equal to a trigonometrical series of the following form

$$\begin{aligned} E - M = a_0 &+ a_1 \cos M + a_2 \cos 2\,M + a_3 \cos 3\,M + \ldots \\ &+ b_1 \sin M + b_2 \sin 2\,M + b_3 \sin 3\,M + \ldots \end{aligned}$$

Then, according to Fourier,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} (E - M)\, dM = \frac{1}{2\pi}\int_0^{2\pi} e \sin E\, dM = 0,$$

because, in this integral, the elements from $M = 0$ to $M = \pi$ destroy those from $M = \pi$ to $M = 2\pi$. In the same way

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (E - M)\, dM \cos nM = 0.$$

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* *Untersuchungen des Theils der planetarischen Storungen, welcher aus der Bewegung der Sonne entsteht.* Abhandlungen, Vol. I, page 84.

There remain only the coefficients b_n. They are

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (E - M)\, dM \sin nM.$$

Using integration by parts, we get

$$\int (E - M) \sin nM\, dM = -\frac{\cos nM}{n}(E - M) + \frac{1}{n}\int (dE - dM) \cos nM.$$

If the integration is from 0 to 2π, the term containing $(E - M)$ vanishes, and we have

$$b_n = \frac{1}{n\pi}\int_0^{2\pi} (dE - dM) \cos nM = \frac{1}{n\pi}\int_0^{2\pi} dE \cos nM$$

$$= \frac{1}{n\pi}\int_0^{2\pi} dE \cos n\,(E - e \sin E).$$

Making for brevity $n\,e = x$,

$$b_n = \frac{1}{n\pi}\int_0^{2\pi} dE \cos\,(nE - x \sin E)$$

$$= \frac{1}{n}\left[\frac{1}{\pi}\int_0^{2\pi} \cos nE \cos (x \sin E)\, dE + \frac{1}{\pi}\int_0^{2\pi} \sin nE \sin\,(x \sin E)\, dE\right].$$

The terms in the brackets are by Fourier's theorem equal to the coefficients of $\cos nE$, or $\sin nE$, in the development of $\cos\,(x \sin E)$, or $\sin\,(x \sin E)$ into a trigonometrical series. The development is most easily performed by using exponentials. If

$$[e]^{iE} = z, \quad \text{then}$$

$$\sin E = \frac{z - \dfrac{1}{z}}{2\,i}, \quad \text{hence}$$

$$\cos\,(x \sin E) = \cos\left(\frac{x\left(z - \dfrac{1}{z}\right)}{2i}\right) = \frac{[e]^{\frac{x}{2}\left(z-\frac{1}{z}\right)} + [e]^{-\frac{x}{2}\left(z-\frac{1}{z}\right)}}{2},$$

$$(x \sin E) = \sin\left(\frac{x\left(z - \dfrac{1}{z}\right)}{2i}\right) = \frac{[e]^{\frac{x}{2}\left(z-\frac{1}{z}\right)} - [e]^{-\frac{x}{2}\left(z-\frac{1}{z}\right)}}{2\,i}.$$

But

$$[e]^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = [e]^{\frac{xz}{2}}\,[e]^{-\frac{x}{2z}} = \left(1 + \frac{xz}{2} + \frac{\left(\frac{xz}{2}\right)^2}{2!} + \frac{\left(\frac{xz}{3}\right)^3}{3!} + \ldots\right)$$

$$\left(1 - \frac{x}{2z} + \frac{\left(\frac{x}{2z}\right)^2}{2!} - \frac{\left(\frac{x}{2z}\right)^3}{3!} + \ldots\right).$$

Performing the multiplication and arranging by powers of z, a series is obtained of the form

$$[e]^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_0(x) + z\,J_1(x) + z^2\,J_2(x) + z^3\,J_3(x) + \ldots$$
$$- \frac{J_1(x)}{z} + \frac{J_2(x)}{z^2} - \frac{J_3(x)}{z^3} + \frac{J_4(x)}{z^4} - \ldots,$$

where in general,

$$(14) \quad J_n(x) = \frac{x^n}{2^n\,n!}\left(1 - \frac{x^2}{2\,(2n+2)} + \frac{x^4}{2\times 4\,(2n+2)(2n+4)} - \frac{x^6}{2\times 4\times 6\,(2n+2)(2n+4)(2n+6)} + \ldots\right).$$

Series (14) is called a *Besselian function.* It depends on two quantities, x and n, and possesses many noteworthy properties, of which the most important is that for every value of x it is finite, and its value is between -1 and $+1$. This follows directly from the definition

$$(15) \qquad J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos\,(nE - x \sin E)\,dE,$$

since this integral is always smaller than

$$\int_0^{2\pi} dE = 2\pi.$$

Moreover, the roots of $J_n(x) = 0$ are all real, so that the curve $y = J_n(x)$ is sinuous, like the curve $y = \sin x$, with the difference that the waves for the first curve are always flatter. Finally, it should be noted that it is always easy to write out $J_{n+1}(x)$ from $J_n(x)$ and $J_{n-1}(x)$, so that, by the transcendentals $J_0(x)$ and $J_1(x)$, the others can be determined.

By use of these functions * we get

$$\cos(x \sin E) = 2\left[J_0(x) + J_2(x)\cos 2E + J_4(x)\cos 4E + J_6(x)\cos 6E + \ldots\right],$$

$$\sin(x \sin E) = 2\left[J_1(x)\sin E + J_3(x)\sin 3E + J_5(x)\sin 5E + \ldots\right].$$

Hence $$b_n = \frac{2}{n} J_n(x) = \frac{2}{n} J_n(ne), \text{ and}$$

$$(16) \qquad E - M = 2\left[J_1(e)\sin M + \tfrac{1}{2} J_2(2e)\sin 2M + \tfrac{1}{3} J_3(3e)\sin 3M + \ldots\right].$$

The quantities r, ξ, and η can also be easily expressed with the help of the Besselian transcendentals. Taking

$$r = a(1 - e\cos E) = a_0 + a_1 \cos M + a_2 \cos 2M + a_3 \cos 3M + \ldots + b_1 \sin M + b_2 \sin 2M + b_3 \sin 3M + \ldots,$$

and we have at once $b_1 = b_2 = b_3 = \ldots. = 0$.

Farther,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} (a - ae\cos E)\, dM = a - \frac{ae}{2\pi}\int_0^{2\pi} \cos E\, dM$$

$$= a - \frac{ae}{2\pi}\int_0^{2\pi} \cos E\,(1 - e\cos E)\, dE = a\left(1 + \frac{e^2}{2}\right),$$

* The Besselian functions are important in other problems. Readers who wish farther information about them, will find their principal properties treated of in Todhunter's work, "An Elementary Treatise on Laplace's, Lamé's and Bessel's Functions."

and, for $n > 0$,

$$a_n = \frac{1}{\pi} \int_0^{2\pi} a\,(1 - e \cos E) \cos nM \, dM$$

$$= \frac{1}{\pi} \left[a\,(1 - e \cos E) \frac{\sin nM}{n} \right]_0^{2\pi} - \frac{a\,e}{n} \frac{1}{\pi} \int_0^{2\pi} \sin nM \sin E \, dE$$

$$= -\frac{a\,e}{n\pi} \int_0^{2\pi} \sin n\,(E - e \sin E) \sin E \, dE$$

$$= -\frac{a\,e}{n} \left[\frac{1}{2\pi} \int_0^{2\pi} \cos\,[(n-1)\,E - n\,e \sin E]\, dE \right.$$

$$\left. - \frac{1}{2\pi} \int_0^{2\pi} \cos\,[(n+1)\,E - n\,e \sin E]\, dE \right]$$

$$= \frac{a\,e}{n} \,[J_{n+1}(n\,e) - J_{n-1}(n\,e)].$$

Hence

$$\frac{r}{a} = 1 + \frac{e^2}{2} + e \left[[J_2(e) - J_0(e)] \cos M + \frac{J_3(2e) - J_1(2e)}{2} \cos 2\,M + \ldots \right]. \tag{17}$$

The coordinate η is derived at once from (16), since

$$\frac{\eta}{a\sqrt{1-e^2}} = \sin E = \frac{E - M}{e}.$$

In the same way ξ can be obtained from (17), since

$$\xi = \frac{a\,(1 - e^2) - r}{e}.$$

The development of v is more troublesome, and the reader is referred to the treatment of it by Bessel.*

* *Analytische Auflösung der Keplerschen Aufgabe.* Abhandlungen. Vol. 1, page 27.

The development of $\dfrac{1}{r}$ is simple. For

$$\frac{1}{r} = \frac{1}{a(1 - e\cos E)} = \frac{1}{a}\frac{dE}{dM},$$

and, by (16),

$$(18) \qquad \frac{a}{r} = 1 + 2\,[J_1(e)\cos M + J_2(2e)\cos 2M + J_3(3e)\cos 3M + \ldots].$$

After developing ξ and η, as indicated above, x, y and z may be expressed in functions of the time; for M and t are connected by the equation $M = nt + \epsilon - \pi = \zeta - \pi$. Taking $\zeta = 0$ as the plane of the orbit, the equations (26a) p. 8 become $x = \xi\xi_x + \eta\eta_x$, $y = \xi\xi_y + \eta\eta_y$, and $z = \xi\xi_z + \eta\eta_z$; the coefficients of ξ, η and ζ being given by (18) p. 16. e and i are generally small; hence limiting the approximation to their second powers in the values of x and y and to their first powers in that of z, we get

$$(19)\begin{cases} x = a\Big\{\cos\zeta + \dfrac{e}{2}[-3\cos\pi + \cos(2\zeta - \pi)] \\ \qquad + \dfrac{e^2}{8}[3\cos(3\zeta - 2\pi) - 4\cos\zeta + \cos(2\pi - \zeta)] \\ \qquad + \dfrac{i^2}{4}[\cos(2\Omega - \zeta) - \cos\zeta]\Big\}, \\ y = a\Big\{\sin\zeta + \dfrac{e}{2}[-3\sin\pi + \sin(2\zeta - \pi)] \\ \qquad + \dfrac{e^2}{8}[3\sin(3\zeta - 2\pi) - 4\sin\zeta + \sin(2\pi - \zeta)] \\ \qquad + \dfrac{i^2}{4}[\sin(2\Omega - \zeta) - \sin\zeta]\Big\}, \\ z = a\,i\sin(\zeta - \Omega). \end{cases}$$

Finally, from (17) to the same degree of approximation,

$$(20) \qquad r = a\left\{1 - e\cos(\zeta - \pi) + \frac{e^2}{2}[1 - \cos 2(\zeta - \pi)]\right\}$$

and thence

$$(21) \qquad r^2 = a^2\left\{1 - 2e\cos(\zeta - \pi) + \frac{e^2}{2}[3 - \cos 2(\zeta - \pi)]\right\},$$

which will be used later.

Formulas (19) can be completed by taking into account the higher powers of the excentricity and inclination, but we shall

content ourselves with deducing a result which is of great importance in what follows.

From (8) and (10) it appears that when $k\,a\,e^{a}\cos\lambda M$, or $k\,a\,e^{a}\sin\lambda M$, is any term in the development of ξ and η, and λ is taken as positive, then we always have $\alpha - \lambda$ either a positive odd number or $= -1$.

The terms for ξ in which $\alpha - \lambda = -1$ are the following:

$$(22)\qquad a\left(\cos M + \frac{e}{2}\cos 2M + \frac{e^2}{2!\,2^2}3\cos 3M\right.$$
$$\left. + \frac{e^3}{3!\,2^3}4^2\cos 4M + \frac{e^4}{4!\,2^4}5^3\cos 5M + \ldots\right),$$

and the corresponding ones for η are

$$(23)\qquad a\left(\sin M + \frac{e}{2}\sin 2M + \frac{e^2}{2!\,2^2}3\sin 3M\right.$$
$$\left. + \frac{e^3}{3!\,2^3}4^2\sin 4M + \ldots\right).$$

By equations (26a) p. 8 and (18) p. 16, the coefficient of ξ in x is

$$\begin{aligned}\xi_x &= \cos(\pi - \Omega)\cos\Omega - \sin(\pi - \Omega)\sin\Omega\cos i\\ &= \cos\pi + \cos(2\Omega - \pi)\sin^2\tfrac{1}{2}i - \cos\pi\sin^2\tfrac{1}{2}i,\end{aligned}$$

and that for η,

$$\begin{aligned}\eta_x &= -\sin(\pi - \Omega)\cos\Omega - \cos(\pi - \Omega)\cos\Omega\cos i\\ &= -\sin\pi + \sin(2\Omega - \pi)\sin^2\tfrac{1}{2}i + \sin\pi\sin^2\tfrac{1}{2}i.\end{aligned}$$

Hence, taking for ξ a term of the form $k\,a\,e^{a}\cos\lambda M = k\,a\,e^{a}$ $\cos\lambda(\zeta - \pi)$, putting it in (26a) p. 8 developing $\sin^2\frac{1}{2}i$ in a series on the ascending powers of i, and using the formula $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$, we get, in the expression for x, terms of the form

$$(24)\quad\left\{\begin{array}{l} k\,a\,e^{a}\,i^{\beta}\cos(\lambda\zeta + \gamma\pi + \delta\Omega),\\ \qquad\text{where in every case}\\ \lambda + \gamma + \delta = +1, \text{ and } \delta = 0 \text{ or } \delta = +2,\\ \qquad\text{and, also,}\\ \alpha + \beta - [\lambda] \text{ is odd and positive, except when } \beta = 0, \text{ in}\\ \text{which case } \alpha - [\lambda] \text{ may also} = -1.\end{array}\right.$$

Here $[\lambda]$ is the absolute value of the quantity λ.

Terms of the same form can be obtained for η, so that in x only the form given in (24) can be found. With the aid of (22) and (23), we can at once write out the terms for which $\alpha - [\lambda] = -1$. They are

$$(25) \qquad a\left[\cos\zeta + \frac{e}{2}\cos(2\zeta - \pi) + \frac{e^2}{2!\,2^2}3\cos(3\zeta - 2\pi) + \frac{e^3}{3!\,2^3}4^2\cos(4\zeta - 3\pi) + \ldots\right].$$

Exactly the same results can be obtained for y, except that cosine must be replaced by sine. Finally, it follows from the formula for z, (26a) p. 8, that in z there are only terms of the form $kae^\alpha i^\beta \sin(\lambda\zeta + \gamma\pi + \delta\Omega)$, and for these $\lambda + \gamma + \delta = 0$, $\delta = \pm 1$, and $\alpha + \beta - [\lambda]$ is either even and positive or $= 0$.

We will bring these results together.

Let the development x be

$$(26) \qquad x = \Sigma k\,a\,e^\alpha\, i^\beta \cos(\lambda\zeta + \gamma\pi + \delta\Omega).$$

Then the development of y is

$$(27) \qquad y = \Sigma k\,a\,e^\alpha\, i^\beta \sin(\lambda\zeta + \gamma\pi + \delta\Omega)$$

with the same coefficient k, and at the same time

(28) $\begin{cases} \lambda + \gamma + \delta = +1,\ \delta = 0 \text{ or } +2, \text{ so that } \lambda + \gamma = \pm 1. \\ \alpha + \beta - [\lambda] \text{ is odd and positive, or when } \beta = 0, \text{ in the} \\ \quad \text{limiting case, the first sum} = -1. \end{cases}$

Let the development of z be

$$(29) \qquad z = \Sigma k'\, e^\alpha\, i^\beta \sin(\lambda\zeta + \gamma\pi + \delta\Omega): \text{ then}$$

$$(30) \qquad \begin{cases} \lambda + \gamma + \delta = 0,\ \delta = \pm 1, \text{ and} \\ \alpha + \beta - [\lambda] \text{ is even and positive, or} = 0. \end{cases}$$

At the same time in all three cases,

$$(31) \qquad \begin{cases} \alpha - [\gamma] \text{ even, positive, or } 0, \\ \beta - [\delta] \text{ even, positive, or } 0. \end{cases}$$

The five quantities $\alpha, \beta, \lambda, \gamma, \delta$ can have all possible integral values, positive or negative, if the conditions (28) to (31) are fulfilled, and it only remains to determine the coefficients k and k' which are pure numbers. This development is the more expeditious, in so far as all the elements a, e, i, π, Ω, except ε

(which, however, appears in $\zeta = nt + \varepsilon$) are analytically expressed. If, however, the order is that of multiples of ζ, the development can be expressed as follows:

$$x = \Sigma(l \cos \lambda\zeta + m \sin \lambda\zeta),$$
$$y = \Sigma(l \sin \lambda\zeta - m \cos \lambda\zeta),$$
$$z = \Sigma(l' \sin \lambda\zeta + m' \cos \lambda\zeta).$$

5. Historical Notes on the Preceding Sections.

Newton's law of gravitation is the source of our explanations of the motions of the heavenly bodies. Its correctness is confirmed by attentive and unbiassed observation, while the development of our knowledge of it is due to the inventive audacity and persistence of mathematical analysis.

Observation shows that the stars apparently describe circles, their common center being near the pole-star. This is not the place to give the reasons which led Copernicus to conclude that these motions were only the reflex of a single rotation, and, hence to propose the hypothesis that *The earth is a body hanging free in space rotating uniformly about an axis which passes through its poles.*

Aside from this motion, the stars remain at rest,—at least for the unaided eye and for the duration of a human life. An exception to this fixity is found in a few bodies, the sun, moon, planets, comets and meteorites. Of these, the first three move with some uniformity among the fixed stars, while the others show striking irregularities which were noticed in very early times.

On the hypothesis of a motionless earth, Ptolemy (Claudius) undertook in the second century after Christ, in his work which was later called the *Almagest,* to reduce the observed motions to circles, or rather to combinations of circular motions. Though this assumption is now rejected, it must be admitted that it was in accord with the ruder observations of antiquity. As observations continued and irregularities multiplied, they were obliged to combine a large number of circles into cycloids, until at last the entire wheelwork became so complicated that only a few could grasp it all, and to these few was attributed an

extraordinary degree of learning. This circumstance, together with the religious respect paid in those days to authority, permitted the *Almagest* to appear as undeniable truth, and any explanation was right or wrong, according as it agreed with that authority or not. The hypothesis of the earth's rotation, which had been already proposed by the Pythagorians and Aristarchus, fell into almost entire oblivion, until Copernicus (1473–1543) appeared and shattered a structure which had existed for thirteen centuries. Copernicus's system was given to the world in a work which appeared just before his death and which was entitled *De Orbium Cœlestium Revolutionibus, libri VI.* By assuming that the earth rotated on its axis and also moved in a circle about the sun, he reached the conclusion that the orbits of the planets were circles, and he explained the observed irregularities by the fact that we observe these circles from a moving station of observation. He adhered to the theory of circular motion, but assumed that the center did not exactly coincide with the sun.

The law of gravitation was soon introduced and was gradually brought to a high degree of perfection by the successive labors of a few great men, each one of whom took up the work where it was left by his predecessor. Tycho Brahe (1546–1601) took new observations with perfected instruments, especially on the planet Mars. Relying on these observations and explaining them by the Copernican theory, Kepler (1571–1630) succeeded gradually in evolving the three laws which are called by his name, and which he published in his two works *De motibus stellae Martis* and *Harmonices Mundi.* He even expressed clear views on universal gravitation, but he was overcome by work and vexations and left the crowning of his labors to a successor who, instead of his method of bold and happy speculation, employed a powerful synthesis.

Meantime, Kepler's contemporary, Galileo (1564–1642) awoke mechanics from the sleep of a thousand years, into which it sank after the death of Archimedes. Galileo's work appeared as *Discorsi e dimostrazioni Matematiche intorno a due nuove scienze attenenti alla Meccanica, ed ai movimenti locali.* The year of

his death was that of the birth of Isaac Newton who first conceived the idea of gravity in 1666, and scientifically grounded the law of gravitation in 1686—the latter in a work entitled *Philosophiae Naturalis Principia Mathematica.* It can not lessen Newton's credit that Boulliau had already expressed this law, but only as an assumption without any proof. By the aid of the law of gravitation Newton proved the laws of Kepler and extended their validity to the comets, the bugbears of earlier times.

Newton's surpassing genius appears in a clearer light when we consider the state of science at that time. Mechanics was as yet barely outlined, the Infinitesimal Calculus owed its origin largely to him, and that powerful analysis did not yet exist by the help of which our demonstrations attain such extraordinary simplicity and transparency. Newton's method was, for the most part, synthetic, yet his results can be reached only with considerable difficulty by the analytical methods which now add so much to our facility in the solution of problems.

With the development of the analysis, for which we are especially indebted to Leibnitz and Bernoulli, the subject was simplified and the problem was reduced to the form of three total simultaneous differential equations, the integration of which can be performed in many different ways. The method employed in this book is that invented by Laplace and given in his *Mécanique céléste.* The literature on this subject, and especially that of the derivation of the coordinates by Kepler's equation, is remarkably large. The principal contributors were Euler, Gauss, Lagrange and Bessel.

6. THE PROBLEM OF n BODIES: THE GENERAL INTEGRALS.

If more than two points attract each other under Newton's law of gravitation, we can, on the principle of composition of forces, get the components of the force acting on any point by adding the corresponding components of the forces exerted by the other points on this point. If we designate the points by P_1, P_2, P_3 .... P_n, each point with its proper index, and if we represent the distance between any two points P_λ and P_μ by $r_{\lambda\mu}$, so that

(1) $$r_{\lambda\mu} = \sqrt{(x_\lambda - x_\mu)^2 + (y_\lambda - y_\mu)^2 + (z_\lambda - z_\mu)^2},$$

we get for the components $m_1 \frac{d^2x_1}{dt^2}$, $m_1 \frac{d^2y_1}{dt^2}$, $m_1 \frac{d^2z_1}{dt^2}$ of the first point by (2), § 1, and the parallelogram of forces,

(2) $$* \quad m_1 \frac{d^2x_1}{dt^2} = k^2 \left(m_1 m_2 \frac{x_2 - x_1}{r_{12}^3} + m_1 m_3 \frac{x_3 - x_1}{r_{13}^3} + \dots \right.$$
$$\left. + m_1 m_n \frac{x_n - x_1}{r_{1n}^3} \right).$$

Similar equations for the other points are obtained by simple exchange of subscripts. An exceptionally elegant form can be given to them, if it is remembered that the right-hand members of equations (2) are the partial derivatives of a function of the coordinates with respect to those coordinates whose moving force is in the left-hand members. This function, as the differentiation at once shows, is

(3) $$V = k^2 \left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} \dots + \frac{m_{n-1} m_n}{r_{(n-1)\,n}} \right) = k^2 \sum \frac{m_\lambda m_\mu}{r_{\lambda\mu}}.$$

This function, the existence of which was first shown by Lagrange, is, after the example of Green (1828) and Gauss (1839), called the *Potential* of the forces. It plays an important part in other branches of Mechanics and Physics. Introducing V, equations (2) take the simple form

(4) $$* \quad m_1 \frac{d^2x_\lambda}{dt^2} = \frac{\partial V}{\partial x_\lambda} \qquad (\lambda = 1,\ 2 \dots n).$$

V satisfies several partial differential equations from which by the assistance of (4), integrable total differential equations can be formed. Since V is a function of the distances, it is a function of the differences of the coordinates, and consequently the sum of its partial derivatives with respect to any coordinate, as x, is equal to zero. Therefore

(5) $$* \quad 0 = \sum \frac{\partial V}{\partial x}.$$

Further, by a rotation of amount φ about the axis of x, y and z become

$$y = y' \cos\varphi - z' \sin\varphi,$$
$$z = y' \sin\varphi + z' \cos\varphi,$$

By this transformation V remains unchanged and independent of φ; consequently its partial derivative with respect to φ is equal to zero. Hence

$$0 = \frac{\partial V}{\partial y_1}\frac{\partial y_1}{\partial \varphi} + \frac{\partial V}{\partial z_1}\frac{\partial z_1}{\partial \varphi} + \dots$$
$$= -z_1\frac{\partial V}{\partial y_1} + y_1\frac{\partial V}{\partial z_1} - \dots$$

Or

(6) $$* \quad 0 = \sum\left(y\frac{\partial V}{\partial z} - z\frac{\partial V}{\partial y}\right).$$

From (4) and (5) we get at once

(7) $$* \quad 0 = \sum m\frac{d^2x}{dt^2}.$$

By a single integration these become

(8) $$* \quad \sum m\frac{dx}{dt} = a_x.$$

Finally, by another integration

(9) $$* \quad \Sigma m x = a_x t + a_x',$$

which equations, a_x, a_x', being the six constants of integration, prove the law of the *conservation of the motion of the center of gravity.* For if its coordinates be represented by ξ, η, ζ, and the sum of the masses Σm by M, we have

(10) $$\xi = \frac{\Sigma m x}{M}, \quad \eta = \frac{\Sigma m y}{M}, \quad \zeta = \frac{\Sigma m z}{M},$$

and hence

(11) $$* \quad M\xi = a_x t + a_x'.$$

By elimination of t from (11) we obtain two independent equations of the first degree between ξ, η, ζ. The center of gravity must, then, move in a straight line with constant velocity. It is

$$= \sqrt{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2} = \frac{\sqrt{a_x^2 + a_y^2 + a_z^2}}{M}.$$

If $a_x = a_y = a_z = 0$, the center of gravity is fixed.

The sun and its planets and comets form a closed system

whose members gravitate toward each other, while the outer forces, due to the fixed stars, are very small, on account of the great distances of the latter. The center of gravity of the solar system, falling generally within the sun, will move, if it moves at all, in a line so nearly rectilinear that its curvature will be sensible only after the lapse of a long period. This motion is the *proper motion of the sun* or of the solar system. Gauss concluded from the proper motions of the stars that, with a high degree of probability, the sun moved toward the constellation of Hercules a result, which was first found by Sir W. Herchel. Later spectroscopic observations have confirmed this result.

From (4) and (6) follow at once,

$$(12) \qquad * \quad 0 = \sum m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right).$$

The equations are integrable at once, and give

$$(13) \qquad * \quad 0 = \sum m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right).$$

Equations (13) are the integrals of area, because, in accordance with § 1, page 4, they express the fact that *The sums of the products of the masses and the projections of the areas described by the radius vector vary as the times.*

From (4) by multiplying each equation by its $\frac{dx_1}{dt}, \frac{dx_2}{dt} \ldots,$ $\frac{dy_1}{dt}, \frac{dy_2}{dt} \ldots, \frac{dz_1}{dt}, \frac{dz_2}{dt} \ldots,$ and adding we get

$$\sum m\left(\frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{d^2y}{dt^2}\frac{dy}{dt} + \frac{d^2z}{dt^2}\frac{dz}{dt}\right)$$
$$= \sum\left(\frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}\right).$$

By integration, this becomes

$$(14) \qquad \frac{1}{2}\sum m\left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) = V + C.$$

This is the equation of the kinetic energy for this particular case. It expresses the fact that one-half the sum of the products of the masses into the velocities differs from the potential by a constant only.

It should be noted that the kinetic energy, and therefore, some velocity, can become infinite, according to (14), only when V is infinite, or when one or more distances of the points $= 0$. In that case, however, the differential equations (4) lose their significance because some of the right-hand members become infinite. The rules of the Calculus are applicable only up to the instant of collision; hence, in this entire discussion, there is a preliminary assumption that no collision occurs, at least none during the time selected.

The integrals (9), (13), (14) are the only ones so far discovered for the general case, and it has been lately shown by Professor Bruno that no other algebraic, nor Abelian integrals exist. While the investigations of the mathematicians have, as will be shown in later paragraphs, brought to light relations of the highest interest, there is, nevertheless, reason to think that the analytical functions so far introduced do not suffice for the solution of the problem. Systems of differential equations are generally not submissive to integration; their very generality gives them an obscurity which is not likely to yield to any other than a complete general treatment.

As the center of gravity plays so important a part, it is very often advisable to refer the motions to it, by employing a moving system of coordinates, whose origin is at this center and whose axes are parallel to the original axes. If the coordinates in this system are represented by primes, and those of the center of gravity referred to the stationary system by ξ, η, ζ, then

$$(15) \qquad * \quad x_1 = x_1' + \xi,$$

and so forth. If these are introduced into (4), ξ, η, ζ, disappear at once from the right-hand members, and they disappear from the left because $\frac{d^2\xi}{dt^2} = \frac{d^2\eta}{dt^2} = \frac{d^2\zeta}{dt^2} = 0$. In other words:

The law of gravitation applies also to the relative motions about the moving center of gravity.

The entire motion of the system, then, consists of two parts: a motion of the system as a whole, represented by that of the center of gravity, and an internal motion about the latter.

The integrals (9), (13), (14) must remain significant for the primed coordinates, but the constants of integration must have other values, which may be represented by b's. Naturally, then

$$b_x = b_x' = b_y = b_y' = b_z = b_z' = 0.$$

Further, from (15), if we put $\Sigma m = M$

$$\sum m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) = \Sigma m\left(y'\frac{dz'}{dt} - z'\frac{dy'}{dt}\right)$$

$$+ \eta\,\Sigma m\frac{dz'}{dt} - \zeta\,\Sigma m\frac{dy'}{dt} - \frac{d\eta}{dt}\Sigma m z'$$

$$+ \frac{d\zeta}{dt}\Sigma m y' + \mathrm{M}\left(\eta\frac{d\zeta}{dt} - \zeta\frac{d\eta}{dt}\right),$$

and, since $\Sigma m x' = \Sigma m y' = \Sigma m z' = 0$, and by (11),

$$\eta\frac{d\zeta}{dt} - \zeta\frac{d\eta}{dt} = \frac{a_y' a_z - a_z' a_y}{M^2},$$

we get

$$(16)^* \quad \sum m\left(y'\frac{dz'}{dt} - z'\frac{dy'}{dt}\right) = C_x' = C_x - \frac{a_y' a_z - a_z' a_y}{M}.$$

Finally,

$$(17) \quad \sum\frac{m}{2}\frac{dx^2 + dy^2 + dz^2}{dt^2} = \sum\frac{m}{2}\frac{dx'^2 + dy'^2 + dz'^2}{dt^2}$$

$$+ \frac{M}{2}\frac{d\xi^2 + d\eta^2 + d\zeta^2}{dt^2}.$$

This equation shows that the kinetic energy of a system of bodies is equal to the kinetic energy due to the motion of the bodies about their center of gravity, plus the kinetic energy due to a motion, equal to that of the center of gravity, of a body whose mass equals the combined masses of the bodies.

From (17) we get, therefore,

$$(18) \quad \sum\frac{1}{2}m\left(\frac{dx'^2 + dy'^2 + dz'^2}{dt^2}\right) = V' + C' = V + C$$

$$- \frac{1}{2}\frac{a_x^2 + a_y^2 + a_z^2}{M},$$

where V' is the same as V except that the relative coordinates are used.

To equation (17) Lagrange gave a form which admits of a highly interesting conclusion concerning the stability of the system.

Introducing into (4) the primed coordinates, and multiplying the equations in order by x_1', y_1', z_1', etc., and remembering that V is a homogeneous function of the coordinates of the degree -1 and hence, satisfies the equation

$$\sum\left(\frac{\partial V}{\partial x}x + \frac{\partial V}{\partial y}y + \frac{\partial V}{\partial z}z\right) = -V,$$

we get

$$\sum m\left(x'\frac{d^2x'}{dt^2} + y'\frac{d^2y'}{dt^2} + z'\frac{d^2z'}{dt^2}\right) = -V', \tag{19}$$

and, adding twice equation (18) and reducing,

$$\frac{d^2\left[\sum\frac{m}{2}(x'^2 + y'^2 + z'^2)\right]}{dt^2} = V' + 2C'. \tag{20}$$

This equation can be given a form in which the mutual distances of the planets alone appear. It is

$$r_{\lambda\mu}^2 = (x_\lambda' - x_\mu')^2 + (y_\lambda' - y_\mu')^2 + (z_\lambda' - z_\mu')^2 = x_\lambda'^2 + y_\lambda'^2 + z_\lambda'^2 + x_\mu'^2 + y_\mu'^2 + z_\mu'^2 - 2(x_\lambda' x_\mu' + y_\lambda' y_\mu' + z_\lambda' z_\mu').$$

By multiplication by $m_\lambda m_\mu$ and addition,

$$\Sigma m_\lambda m_\mu r_{\lambda\mu}^2 = \Sigma m_\lambda m_\mu[(x_\lambda'^2 + y_\lambda'^2 + z_\lambda'^2) + (x_\mu'^2 + y_\mu'^2 + z_\mu'^2)] - \Sigma 2 m_\lambda m_\mu(x_\lambda' x_\mu' + y_\lambda' y_\mu' + z_\lambda' z_\mu') = M\Sigma m_\lambda(x_\lambda'^2 + y_\lambda'^2 + z_\lambda'^2) - \Sigma[m_\lambda^2(x_\lambda'^2 + y_\lambda'^2 + z_\lambda'^2) + 2 m_\lambda m_\mu(x_\lambda' x_\mu' + y_\lambda' y_\mu' + z_\lambda' z_\mu')].$$

The last sum $= (\Sigma m_\lambda x_\lambda')^2 + (\Sigma m_\lambda y_\lambda')^2 + (\Sigma m_\lambda z_\lambda')^2 = 0$, and (20) becomes

$$\frac{1}{2M}\frac{d^2(\Sigma m_\lambda m_\mu r_{\lambda\mu}^2)}{dt^2} = V' + 2C'. \tag{21}$$

V' is always positive. If C' is also positive the right-hand member is positive and greater than $2C'$. The first derivative of $\Sigma m_\lambda m_\mu r_{\lambda\mu}^2$ will then increase continuously. If it is originally negative, it will at some time $= 0$ and will afterwards be positive, so that in this case $\Sigma m_\lambda m_\mu r_{\lambda\mu}^2$ grows with acceler-

ated velocity to infinity. A distance $r_{\lambda\mu}$ must eventually become infinite, that is one point must at last separate itself from the others to an infinite distance. If by stability of a system is meant the condition that these distances always remain finite (never 0 and never ∞), we can say:

In order that a system may be stable, C' must be neither positive nor zero; that is, the kinetic energy must be less than the potential.

It does not, however, necessarily follow that when C' is negative the system is stable. Only partial stability can be concluded from it since not all the distances can become infinite. For if all points separated infinite distances from each other V' would $= 0$ and the right-hand member of (21) would be negative. $\Sigma m_\lambda m_\mu r_{\lambda\mu}^2$ must finally become smaller, and at least two points approach each other, and this is impossible since at infinity the attraction $= 0$.

The conception of complete stability, as defined above, is of the highest importance for astronomy. The solar system, including only sun and planets, is, in all probability, a stable system. The labors of Laplace and Lagrange and of later astronomers culminate in the problem of stability, and they have shown that this conception can be narrowed down to a great degree when applied to this system. Unfortunately, however, they have not succeeded in giving an entirely rigorous proof, nor, in general, in fixing the conditions of stability. At present it can only be said that our solar system may be stable, and, with great probability, is stable without limit as to time.

Finally, we will add a paragraph on the conception of the invariable plane, as represented by Laplace for our solar system. The expressions

$$y'\frac{dz'}{dt} - z'\frac{dy'}{dt}, \quad z'\frac{dx'}{dt} - x'\frac{dz'}{dt}, \quad x'\frac{dy'}{dt} - y'\frac{dx'}{dt}$$

have the remarkable property that, with a rotation of the system of coordinates, they are transformed in the same manner as the coordinates. Taking the familiar transformation formulas

$$(22) \qquad * \quad x'' = x'\alpha_x + y'\beta_x + z'\gamma_x,$$

and representing the first set by A', B', C', the transformed ones by A'', B'', C'', we get from (22)

$$A'' = A' (\beta_y \gamma_z - \gamma_y \beta_z) + B' (\gamma_y \alpha_z - \alpha_y \gamma_z) + C' (\alpha_y \beta_z - \beta_y \alpha_z),$$

or, by known relations between the nine coefficients,

$$(23) \qquad * \quad A'' = A' \alpha_x + B' \beta_x + C' \gamma_x.$$

If we represent by C_x'', C_y'', C_z'' the constants which replace C_x', C_y', C_z' of equations (16), we get similarly,

$$(24) \qquad * \quad C_x'' = C_x' \alpha_x + C_y' \beta_x + C_z' \gamma_x.$$

The constants of integration are, then, transformed in the same way as the coordinates. If a new system is selected such that $C_y'' = C_z'' = 0$, then two equations fix the x'' axis completely, but not those of y'' and z''. For from them is derived,

$$C_x' : C_y' : C_z' = \beta_y \gamma_z - \gamma_y \beta_z : \gamma_y \alpha_z - \alpha_y \gamma_z : \alpha_y \beta_z - \alpha_z \beta_y = \alpha_x : \beta_x : \gamma_x,$$

and hence,

$$(25) \begin{cases} \alpha_x = \dfrac{C_x'}{\sqrt{C_x'^2 + C_y'^2 + C_z'^2}}, \quad \beta_x = \dfrac{C_y'}{\sqrt{C_x'^2 + C_y'^2 + C_z'^2}}, \\ \gamma_x = \dfrac{C_3'}{\sqrt{C_x'^2 + C_y'^2 + C_z'^2}}, \text{ and} \end{cases}$$

$$(26) \qquad C_x'' = \sqrt{C_x'^2 + C_y'^2 + C_z'^2}.$$

The y'' z'' plane, thus determined, is Laplace's *invariable plane.* The sum of the areas projected on it is a maximum, while for every x'' plane it is zero. The invariable plane makes but a small angle with the ecliptic, and has the exceptional advantage over the latter, as well as over that of the equator, to which are generally referred the spherical coordinates of the stars, that it is absolutely invariable, while in the course of thousands of years the ecliptic and the equator change their position in space to a notable amount. A proposal to use the invariable plane is, however, for the present illusory, because it can only be fixed after an absolutely exact knowledge of the masses of the planets, and, besides, all astronomical tables now employ ecliptic or equatorial coordinates.

If at any moment all points are in one plane and all the velocities in the same plane, they will always remain in this

plane, and it becomes the invariable plane of the system. This is approximately the case with the solar system.

If $C_x' = C_y' = C_z' = 0$, every plane is invariable. This would be the case, for instance, if all the points were at rest at a given instant, or if they were all in one straight line and their motion was confined to it.

The motions in the solar system are referred by astronomers, not to the center of gravity, but to the center of the sun. This requires some modification in the formulas which will now be given.

Let the Gaussian constant $k = 1$. Let m_1 be the mass of the first planet, x_1, y_1, z_1, its coordinates with reference to the sun, x_1', y_1', z_1', these with reference to the center of gravity of the system, etc. Let ξ, η, ζ be the coordinates of the sun referred to the center of gravity, and let its mass be M. The potential V' has the value

$$V' = M\sum\frac{m_\lambda}{\sqrt{(x_\lambda' - \xi)^2 + (y_\lambda' - \eta)^2 + (z_\lambda' - \zeta)^2}}$$

$$+ \sum\frac{m_\lambda\, m_\mu}{\sqrt{(x_\lambda' - x_\mu')^2 + (y_\lambda' - y_\mu')^2 + (z_\lambda' - z_\mu')^2}}.$$

The equations of motion become therefore

$$* \quad M\frac{d^2\xi}{dt^2} = \frac{\partial V'}{\partial \xi}, \quad * \quad m_1\frac{d^2x_1'}{dt^2} = \frac{\partial V'}{\partial x_1'} \;\ldots\ldots$$

But,

(27) $\qquad x_1 = x_1' - \xi,\;\; y_1 = y_1' - \eta,\;\; z_1 = z_1' - \zeta$, etc.,

and if we put

$$V = \sum\frac{m_\lambda\, m_\mu}{\sqrt{(x_\lambda' - x_\mu')^2 + (y_\lambda' - y_\mu')^2 + (z_\lambda' - z_\lambda')^2}}$$

$$= \sum\frac{m_\lambda\, m_\mu}{\sqrt{(x_\lambda - x_\mu)^2 + (y_\lambda - y_\mu)^2 + (z_\lambda - z_\mu)^2}},$$

$$\sqrt{x_\lambda^2 + y_\lambda^2 + z_\lambda^2} = r_\lambda,$$

$$M + m_\lambda = \mu_\lambda,$$

we get

$$\frac{d^2x_1}{dt^2} = \frac{d^2x_1'}{dt^2} - \frac{d^2\xi}{dt^2} = -m_1\frac{x_1}{r_1^3} + \frac{1}{m_1}\frac{\partial V}{\partial x_1}$$

$$- m_2\frac{x_2}{r_2^3} - m_3\frac{x_3}{r_3^3} \ldots\ldots \text{ etc.}$$

Introducing finally,

$$R_\lambda = \frac{1}{m_\lambda}V - m_1\frac{x_\lambda x_1 + y_\lambda y_1 + z_\lambda z_1}{r_1^3} - m_2\frac{x_\lambda x_2 + y_\lambda y_2 + z_\lambda z_2}{r_2^3}\ldots$$

$$= \frac{1}{m_\lambda}V - \sum m_\mu\frac{x_\lambda x_\mu + y_\lambda y_\mu + z_\lambda z_\mu}{r_\mu^3},$$

where μ may be any number not λ, the equations of motion take the form

$$(28)\ * \quad \frac{d^2x_1}{dt^2} = -\frac{\mu_1 x_1}{r_1^3} + \frac{\partial R_1}{\partial x_1},\quad \frac{d^2x_2}{dt^2} = -\frac{\mu_2 x_2}{r_2^3} + \frac{\partial R_2}{\partial x_2}\ \ldots\ldots$$

The qualities R_1, R_2 ... are called perturbing functions, for reasons which appear later. They depend only on the masses of the planets and on their distances from each other and from the sun. When x_1, y_1, z_1, etc., have been determined by (27), ξ, η, ζ are found by the equation

$$M\xi + \Sigma mx' = M\xi + \Sigma m\,(x+\xi) = 0.$$

Hence,

$$(29)\quad * \begin{cases} \xi = -\dfrac{\Sigma\, mx}{H + \Sigma\, m}, \\ \text{and consequently} \\ x_1' = x_1 - \dfrac{\Sigma\, mx}{M + \Sigma\, m}, \\ \text{etc.} \end{cases}$$

The integral equations (16) and (17) also take on a somewhat different form. They become

$$(30)\quad \begin{cases} \sum m\left(y\dfrac{dz}{dt} - z\dfrac{dy}{dt}\right) + k\left(\Sigma\, my\ \Sigma\, m\dfrac{dz}{dt}\right. \\ \qquad\qquad \left. - \Sigma\, mz\ \Sigma\, m\dfrac{dy}{dt}\right) = C_1', \\ \sum m\left(z\dfrac{dx}{dt} - x\dfrac{dz}{dt}\right) + k\left(\Sigma\, mz\ \Sigma\, m\dfrac{dx}{dt}\right. \end{cases}$$

$$
(30) \quad \begin{cases}
\qquad\qquad - \Sigma mx \, \Sigma m \dfrac{dz}{dt}\Big) = C_2', \\
\sum m \left(x \dfrac{dy}{dt} - y \dfrac{dx}{dt} \right) + k \Big(\Sigma mx \, \Sigma m \dfrac{dy}{dt} \\
\qquad\qquad - \Sigma my \, \Sigma m \dfrac{dx}{dt} \Big) = C_3', \\
k = \dfrac{-1}{M + \Sigma m},
\end{cases}
$$

and

$$
\sum \frac{m}{2} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right]
$$

$$
+ \frac{k}{2} \left[\left(\Sigma m \frac{dx}{dt} \right)^2 + \left(\Sigma m \frac{dy}{dt} \right)^2 + \left(\Sigma m \frac{dz}{dt} \right)^2 \right] - V' = C'.
$$

7. Problem of Three Bodies.

Three bodies only will now be assumed, not because the results obtained for three can be extended to a larger number, but because the formulas obtained with this limitation present an extraordinary symmetry. No new integrals will be afforded, as has already been shown. It will appear, however, that by a skillful and elegant use of the integrals already found, the problem can be notably simplified. This is especially the case when the development is reduced to the parts which are independent of the system of coordinates, making what Hesse calls the reduced problem of three bodies.

The system of nine differential equations are, in this case,

$$
(1) \qquad * \quad m_i \frac{d^2 x_i}{dt^2} = \frac{\partial V}{\partial x_i}, \quad (i = 1, 2, 3),
$$

in which

$$
(2) \qquad V = \frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_3 m_1}{r_{31}} \text{ and}
$$

$$
(3) \qquad r_{\lambda\mu} = \sqrt{(x_\lambda - x_\mu)^2 + (y_\lambda - y_\mu)^2 + (z_\lambda - z_\mu)^2}.
$$

This is a simultaneous system and requires for its solution eighteen integrations, of which ten have already been performed, —six for the law of conservation of motion of the center of

gravity, three for the areal law, and one for the law of conservation of energy. Aside from these results, the system can be reduced to one with only two variables x_i and t, for instance.

If the first of the equations (1) is differentiated with t as the independent variable, we get

$$
\begin{aligned}
(4) \qquad m_i \frac{d^3x_i}{dt^3} = & \frac{\partial^2 V}{\partial x_i \partial x_1} u_1 + \frac{\partial^2 V}{\partial x_i \partial y_1} v_1 + \frac{\partial^2 V}{\partial x_i \partial z_1} w_1 \\
& + \frac{\partial^2 V}{\partial x_i \partial x_2} u_2 + \frac{\partial^2 V}{\partial x_i \partial y_2} v_2 + \frac{\partial^2 V}{\partial x_i \partial z_2} w_2 \\
& + \frac{\partial^2 V}{\partial x_i \partial x_3} u_3 + \frac{\partial^2 V}{\partial x_i \partial y_3} v_3 + \frac{\partial^2 V}{\partial x_i \partial z_3} w_3
\end{aligned}
$$

where $u_i = \frac{dx_i}{dt}$, $v_i = \frac{dy_i}{dt}$, $w_i = \frac{dz_i}{dt}$, and it will follow that $\frac{du_i}{dt} = \frac{d^2x_i}{dt^2} \cdots$.

The right-hand member of (4) is a function of the coordinates and their first derivatives. If (4) is again differentiated with respect to t, and the second derivatives are eliminated by (1), $\frac{d^4x_i}{dt^4}$ is also a function of the coordinates and their first derivatives, and it becomes evident that $\frac{d^nx_i}{dt^n}$ is a function of the same quantities. Fixing the attention only on x_i, $\frac{dx_i}{dt}$, $\frac{d^2x_i}{dt^2}$, $\frac{d^3x_i}{dt^3}$, etc., it appears that in these equations there are only sixteen elements, (the eight other coordinates and their first derivatives), which have a perturbing action, and, to eliminate these we must have seventeen equations. The proper way to do this is to express $\frac{d^2x_i}{dt^2}$, $\frac{d^3x_i}{dt^3} \cdots \frac{d^{18}x_i}{dt^{18}}$ as functions of the nine coordinates and their first derivatives and, by means of the seventeen resulting equations, to eliminate the eight remaining coordinates and their derivatives. The final result is an equation of the form

$$
(5) \qquad f\left(x_i, \frac{dx_i}{dt}, \frac{d^2x_i}{dt^2}, \cdots \frac{d^{18}x_i}{dt^{18}}\right) = 0,
$$

that is, a differential equation of the eighteenth order between x_i and t.

When (5) is completely integrated, the problem becomes one of pure elimination, and the previous expressions for $\frac{d^2 x_i}{dt^2} \cdots \frac{d^{18} x_i}{dt^{18}}$ are more than enough to calculate the remaining coordinates and their derivatives.

In the place of a coordinate, any function of the coordinates and their first derivatives may be employed. This will in general lead to a differential equation of the eighteenth order between the function and the time, but in special cases it may lead to an equation of a lower order. Indeed, it may be the case that the equation is of the first order and takes the simple form

$$\frac{d\varphi}{dt} = 0,$$

and this is always the case when φ is an integral of the system.

In general, an elimination of all the variables but two, and even the elimination of a less number, will not be of use because the order of the differential equations rises as the elimination proceeds and their number is decreased.

The fact that equations (1) are independent of the system of coordinates selected gives them a property which was first clearly pointed out by Lagrange in his celebrated paper *Sur le problème des trois corps.* It is that a special system of unknowns can be given, such that with them the problem requires only nine integrations,—nine less than the general system of equations. The unknowns must be so selected that they depend only on the mutual distances of the bodies, and, in this case, they are preferably r_{12}, r_{23}, r_{31}. The problem thus transformed is Hesse's reduced problem of three bodies.

Besides the fact that this reduction reduces the order of the system by nine, there is another and a deeper ground for it. In anv system of moving points, without further hypothesis, there are only changes in the distances and directions, and the latter can always be expressed by the former. It is therefore clear that, aside from all hypothesis, our real knowledge of the motions

of bodies in space can not surpass that given by the reduced problem. The laws of the motions of bodies are deduced from observation and also from hypotheses, which, however simple and credible they may be, are still hypothetical. If we remove the results of the latter from our problem, what remains is the reduced problem.

Observations alone would never have disclosed Newton's law, even if we had been able to measure the distances directly. It is only on the hypothesis that the fixed stars are infinitely distant and at rest that the real changes in direction can be determined. Farther, the principle of conservation of the motion of the center of gravity of our system is an hypothesis without any basis in direct observation, created by Newton out of his principle of action and reaction; and, however great its probability on philosophical and other grounds, it is incapable of a rigorous demonstration.

The reduced problem for two bodies is expressed by (16), § 1

$$\frac{d^2\left(r\,\frac{dr}{dt}\right)}{dt^2} = -\mu\,\frac{r\,\dfrac{dr}{dt}}{r^3}, \tag{6}$$

which is of the third order, while the general problem requires twelve integrations. The problem then is, to get for three bodies the equations corresponding to (6). This has been done by Lagrange in the most elegant manner, and to lay the foundations for it, we will first transform equations (1) by introducing the components of velocity which will be indicated by major letters, thus:

$$* \quad X_i = \frac{dx_i}{dt}. \tag{7}$$

Equations (1) now become

$$* \quad m_i\,\frac{dX_i}{dt} = \frac{\partial V}{\partial x_i},\ (i = 1, 2, 3). \tag{8}$$

If we take x_i and X_i as variables, then (7) and (8) will together form a system of total differential equations which can be used for (1). It will be noticed that by this process the number of these equations is doubled but we now have equations of the

first order only, instead of the second, and this is more convenient for many transformations.

Equations (1) retain their form if, instead of the given conditions, others are introduced by orthogonal transformation in a new fixed system of coordinates, or in a system which moves uniformly relative to the given one. Taking (7) and (8), we see that they retain their form if we put

$$(9) \qquad \begin{cases} * \quad x_i = \alpha_x x_i' + \beta_x y_i' + \gamma_x z_i' + a_x t + b_x, \\ * \quad X_i = \alpha_x X_i + \beta_x Y_i + \gamma_x Z_i + a_x, \end{cases}$$

where α, β, γ are the nine coefficients of an orthogonal transformation, and a and b, any six given quantities. Hence (7) and (8) permit the transformation (9), and if they are together of the 18th order, it follows that, if we consider only such functions of x_i and X_i as permit this transformation, they will be reduced to the $18 - 9 = 9^{\text{th}}$ order. It is clear that we may take r_{12}, r_{23}, r_{31} for such functions, but it is better first to consider all the nine functions, which are independent and by which the others may be expressed, as may easily be shown. A function which permits the transformation above depends only on the differences between the quantities x_i and between the quantities X_i. Putting

$$(10) \qquad \begin{cases} * \quad x_1 - x_2 = x''', \; x_2 - x_3 = x', \; x_3 - x_1 = x'', \\ * \quad X_1 - X_2 = X''', \; X_2 - X_3 = X', \; X_3 - X_1 = X'', \end{cases}$$

then any such function depends only on the primed quantities. The number of these is eighteen, but it immediately reduces to twelve, since

$$(11) \qquad \begin{cases} * \quad x' + x'' + x''' = 0 \\ * \quad X' + X'' + X''' = 0. \end{cases}$$

This reduction would destroy the symmetry and we will therefore retain all the eighteen x', X', etc.

If we define the twenty-four expressions $[\lambda, \mu]$, $[\lambda, \mu_1]$, $[\lambda_1, \mu]$, by the formulas of Hesse,

$$(12) \qquad \begin{cases} [\lambda, \mu] = x^\lambda x^\mu + y^\lambda y^\mu + z^\lambda z^\mu = [\mu, \lambda], \\ [\lambda, \mu_1] = x^\lambda X^\mu + y^\lambda Y^\mu + z^\lambda Z^\mu, \\ [\lambda_1, \mu] = X^\lambda X^\mu + Y^\lambda Y^\mu + Z^\lambda Z^\mu = [\mu, \lambda_1], \end{cases}$$

they will permit the transformation required and it is clear that they may be expressed linearly, by (11), but for symmetry we shall rather take,

$$(13) \qquad \begin{cases} [1,1],\, [2,2],\, [3,3], \text{ or } r_{23}^2,\, r_{31}^2,\, r_{13}^2, \\ [1,1'],\, [2,2'],\, [3,3'], \\ [1',1],\, [2',2],\, [3',3], \end{cases}$$

and

$$(14) \qquad \rho = [1,2'] - [2,1'],$$

or, by (11),

$$\rho = [1,2'] - [2,1'] = [2,3'] - [3,2'] = [3,1'] - [1,3'].$$

Considering the results of (11),

$$(15) \qquad \begin{cases} [1,1] + [1,2] + [1,3] = 0; \\ [1,1'] + [1,2'] + [1,3'] = 0, \text{ etc.,} \end{cases}$$

we find by simple transformations

$$(16) \qquad \begin{cases} [1,2\,] = -\tfrac{1}{2}([1,1\,] + [2,2\,] - [3,3']), \\ [1',2'] = -\tfrac{1}{2}([1',1'] + [2',2'] - [3',3']), \\ [1,2'\,] = -\tfrac{1}{2}([1,1'\,] + [2,2'\,] - [3,3'\,]) + \rho, \\ [2,1'\,] = -\tfrac{1}{2}([1,1'\,] + [2,2'\,] - [3,3'\,]) - \rho, \\ \text{etc.} \end{cases}$$

There are also other simple expressions which permit the transformation (9). These are determinants of the following type:

$$(17) \qquad D = \begin{vmatrix} x',\, y',\, z' \\ x'',\, y'',\, z'' \\ X',\, Y',\, Z' \end{vmatrix}, \text{ etc.}$$

If two such determinants are multiplied by rows, expressions of the form $[\lambda, \mu]$, etc., will be obtained. For instance

$$D^x = \begin{vmatrix} [1,1\,], & [2,1\,], & [1',1\,] \\ [1,2\,], & [2,2\,], & [1',2\,] \\ [1,1'], & [2,1'], & [1',1'] \end{vmatrix}.$$

As nine independent functions, which admit (8), are sufficient for expressing all the others by them, it follows that there must exist one equation containing the nine quantities (13) and ρ. To find this equation, let us take the vanishing determinant

$$\begin{vmatrix} x', & y', & z', & 0 \\ x'', & y'', & z'', & 0 \\ X', & Y', & Z', & 0 \\ X'', & Y'', & Z'', & 0 \end{vmatrix}$$

and form the squares by rows. The square is

$$(18) \qquad \begin{vmatrix} [1, 1\,], & [1, 2\,], & [1, 1'], & [1, 2'] \\ [1, 2\,], & [2, 2\,], & [2, 1'], & [2, 2'] \\ [1, 1'], & [2, 1'], & [1', 1'], & [2', 1'] \\ [1, 2'], & [2, 2'], & [1', 2'], & [2', 2'] \end{vmatrix} = 0,$$

and if we introduce therein expressions (16), we immediately obtain the equation sought, but in an unsymmetrical form. From inspection of (18) we get an equation of the form

$$(19) \qquad A\rho^4 + B\rho^2 + C = 0,$$

and

$$(20) \qquad \rho = \sqrt{\frac{-B + \sqrt{B^2 - 4AC}}{2A}},$$

in which ρ is expressed in terms of the quantities of (13).

We now pass to the formation of the differential equations between the nine quantities of (13) and the time, or rather, since we retain ρ, between (13), ρ_1 and t. To this end we first form the differential equations between

$$* \quad x', \ x'', \ x''',$$

and

$$X', \ X'', \ X''',$$

introducing, for brevity,

$$(21) \qquad \begin{cases} m_1 + m_2 + m_3 = M, \\ \dfrac{1}{r_{13}{}^3} - \dfrac{1}{r_{12}{}^3} = p_1, \\ \dfrac{1}{r_{21}{}^3} - \dfrac{1}{r_{23}{}^3} = p_2, \\ \dfrac{1}{r_{32}{}^3} - \dfrac{1}{r_{31}{}^3} = p_3, \\ p_1 + p_2 + p_3 = 0. \end{cases}$$

From (7) and (8), with the aid of (2), we get them, by a simple transformation, in the form

$$(22)\quad \begin{cases} * \dfrac{dx'}{dt} = m_1 (x''' p_2 - x'' p_3) - M \dfrac{x'}{r_{23}^3}, \\ * \dfrac{dx'}{dt} = X', \end{cases}$$

from which the others may be written by cyclical interchange of the indices. The differentiation of (12) and use of (22) and (16), with simple transformations, gives

$$(23)\quad \begin{cases} \dfrac{d\,[1,1]}{dt} = 2\,[1,1'], \\ \dfrac{d\,[1,1']}{dt} = [1',1'] - \dfrac{m_1}{2}\,(p_1\,([3,3]-[2,2]) \\ \qquad + (p_3 - p_2)\,[1,1]) - \dfrac{M[1,1]}{r_{23}^3}, \\ \dfrac{d\,[1',1']}{dt} = m_1 p_1([3,3'] - [2,2']) + (p_3 - p_2)\,[1,1']) \\ \qquad - \dfrac{2M[1,1']}{r_{23}^3} - \rho\, m_1\, p_1, \end{cases}$$

from which the six others can be written by cyclical interchange of indices.

Finally it follows that

$$(24)\quad \begin{aligned} \frac{d\rho}{dt} &= m_1\,[2,3]\,p_1 + m_2\,[3,1]\,p_2 + m_3\,[1,2]\,p_3 \\ &= -\tfrac{1}{2}([1,1]+[2,2]+[3,3])\,(m_1 p_1 + m_2 p_2 + m_3 p_3) \\ &\quad + m_1 p_1\,[1,1] + m_2 p_2\,[2,2] + m_3 p_3\,[3,3]. \end{aligned}$$

Since $r_{23}^2 = [1, 1]$, etc., equations (23) and (24) are the ten differential equations between $[1, 1]$, $[1, 1']$, $[1', 1']$, etc., ρ and t. The quantities p and r, according to (13) and (21), are directly given by $[1, 1]$, $2, 2]$, $[3, 3]$.

Of the system (23) and (24) there is one integral known, namely, equation (19), and, since this contains no arbitrary constants, it is either a particular or a singular one, but which it has not yet been determined. When ρ is eliminated by means of it, equations (18) represents a complete system of differential equations between $[1, 1]$, $[1, 1']$, $[2', 2']$, etc., and t.

If only the three distances and the time are to be retained in the system, then $[1, 1']$, $[1', 1']$, etc., must be eliminated

from (23). This is easily done, since [1, 1′], etc., are at once derived from the first of these, then [1, 1′] by substitution in the second, and then also ρ by the distances and their first and second derivatives relative to the time. These substituted in the last give a final equation of the form

$$(25) \qquad \tfrac{1}{2}\frac{d^3[1,1]}{dt^3} = \frac{dP_1}{dt} + Q_1 - \rho\, m_1 p_1,$$

where

$$(26) \quad \begin{cases} P_1 = \dfrac{m_1}{2}\left(([3,3]-[2,2])\,p_1 - [1,1]\,(p_3-p_2)\right) - \dfrac{M[1,1]}{r_{23}^{\,3}}, \\ Q_1 = m_1\left(([3,3']-[2,2'])\,p_1 + [1,1']\,(p_3-p_2)\right) - \dfrac{2M[1,1']}{r_{23}^{\,3}}. \end{cases}$$

The expression for ρ is to be substituted in (25). The interchange of indices will then give the two other differential equations of the third order.

From these an integrable equation can be formed for, from (26)

$$\frac{Q_1}{m_1} + \frac{Q_2}{m_2} + \frac{Q_3}{m_3} = -\,2\,\frac{d}{dt}\left(\frac{P_1}{m_1} + \frac{P_2}{m_2} + \frac{P_3}{m_3}\right).$$

Hence, from (25) and its analgues, we get

$$\tfrac{1}{2}\frac{d^3}{dt^3}\left(m_2 m_3 r_{23}^{\,2} + m_3 m_1 r_{31}^{\,2} + m_1 m_2 r_{12}^{\,2}\right)$$

$$= -\frac{dM}{dt}\left(\frac{P_1}{m_1} + \frac{P_2}{m_2} + \frac{P_3}{m_3}\right) m_1 m_2 m_3 = \frac{d}{dt}\,(MV).$$

Then, by integration,

$$(27) \quad \tfrac{1}{2}\frac{d^2}{dt^2}\left(m_2 m_3 r_{23}^{\,2} + m_3 m_1 r_{31}^{\,2} + m_1 m_2 r_{12}^{\,2}\right) = MV + C,$$

or equation (21), § 6.

If equations (25) are multiplied, in order, by $\dfrac{1}{m_1 r_{23}^{\,3}}$, $\dfrac{1}{m_2 r_{31}^{\,3}}$, $\dfrac{1}{m_3 r_{12}^{\,3}}$, and the products added, the ρ disappears. It follows that

$$0 = \left[\tfrac{1}{2}\frac{d^3[1,1]}{dt^3} - \frac{dP_1}{dt} - Q_1\right]\frac{1}{m_1 r_{23}^{\,3}}$$

$$(28)\qquad + \left[\tfrac{1}{2}\frac{d^3[2,2]}{dt^3} - \frac{dP_2}{dt} - Q_2\right]\frac{1}{m_2 r_{31}{}^3}$$
$$+ \left[\tfrac{1}{2}\frac{d^3[3,3]}{dt^3} - \frac{dP_3}{dt} - Q_3\right]\frac{1}{m_3 r_{12}{}^3}.$$

Besides the integral (27), the equations (25) have another, a combination of the surface integrals, found by Lagrange.

$$(29)\qquad \begin{cases} \dfrac{(\eta,\zeta)_1}{m_1} + \dfrac{(\eta,\zeta)_2}{m_2} + \dfrac{(\eta,\zeta)_3}{m_3} = a, \\ \dfrac{(\zeta,\xi)_1}{m_1} + \dfrac{(\zeta,\xi)_2}{m_2} + \dfrac{(\zeta,\xi)_3}{m_3} = b, \\ \dfrac{(\xi,\eta)_1}{m_1} + \dfrac{(\xi,\eta)_2}{m_2} + \dfrac{(\xi,\eta)_3}{m_3} = c, \end{cases}$$

where, for brevity,

$$(30)\qquad \eta_\lambda \frac{d\zeta_\lambda}{dt} - \zeta_\lambda \frac{d\eta_\lambda}{dt} = (\eta,\zeta)_\lambda, \text{ etc.}$$

The combination given by the new integral is the sum of the squares of the areal integrals. For

$$a^2 + b^2 + c^2 = \frac{(\eta,\zeta)_1{}^2 + (\zeta,\xi)_1{}^2 + (\xi,\eta)_1{}^2}{m_1{}^2}$$
$$+ 2\,\frac{(\eta,\zeta)_1(\eta,\zeta)_2 + (\zeta,\xi)_1(\zeta,\xi)_2 + (\xi,\eta)_1(\xi,\eta)_2}{m_1 m_2} + \ldots$$

But, by a known proposition of determinants,

$$(\eta,\zeta)_1{}^2 + (\zeta,\xi)_1{}^2 + (\xi,\eta)_1{}^2 = [1,1]\,[1',1'] - [1,1']^2$$
$$= [1,1]\,(\tfrac{1}{2}[1,1]'' - P_1) - (\tfrac{1}{2}[1,1]')^2,$$
$$(\eta,\zeta)_1(\eta,\zeta)_2 + \ldots = [1,2]\,[1',2'] - [1,2']\,[1',2]$$
$$= [1,2]\left(\tfrac{1}{2}[1,2]'' + \frac{P_1 + P_2 - P_3}{2}\right) - ((\tfrac{1}{2}[1,2]')^2 - \rho^2),$$

etc. Consequently

$$(31)\quad a^2 + b^2 + c^2 = \frac{[1,1]\,(\tfrac{1}{2}[1,1]'' - P_1) - (\tfrac{1}{2}[1,1]')^2}{m_2}$$
$$+ 2\,\frac{[1,2]\left(\tfrac{1}{2}[1,2]'' + \dfrac{P_1 + P_2 - P_3}{2}\right) - ((\tfrac{1}{2}[1,2]')^2 - \rho^2)}{m_1 m_2}$$
$$+ \ldots.$$

And (31) is the integral sought, when for ρ its value in (20) is

taken. The system is reduced to the seventh order and consists of

1. A differential equation of the third order (28),
2. " " " " second " (29),
3. " " " " second " (31).

If t is eliminated from these equations, the order is reduced to the sixth, and this is the extreme of reduction reached to the present time.

When the reduced problem shall have been completely integrated, the final solution of the entire problem will, as shown by Lagrange in his celebrated memoir, depend on a single quadrature.

If equations (29) are multiplied in order by ξ_1, η_1, ζ_1, we get, by appropriate addition,

$$(32)\qquad \begin{vmatrix} \xi_2, & \xi_3, & \dfrac{d\xi_2}{m_2\,dt} - \dfrac{d\xi_3}{m_3\,dt} \\ \eta_2, & \eta_3, & \dfrac{d\eta_2}{m_1\,dt} - \dfrac{d\eta_3}{m_3\,dt} \\ \zeta_2, & \zeta_3, & \dfrac{d\zeta_2}{m_2\,dt} - \dfrac{d\zeta_3}{m_3\,dt} \end{vmatrix} = a\xi_1 + b\eta_1 + c\zeta_1.$$

The left-hand member is a function depending on the distances. If it is squared, by compounding the columns, and the previous formulas are used, the desired form is at once obtained. In the same way the expressions $a\xi_2 + b\eta_2 + c\zeta_3$, and $a\xi_3 + b\eta_3 + c\zeta_3$ may be determined. If ξ, η are taken in the invariable plane, $a = 0$, and $b = 0$, and from (32) and the corresponding equations ζ_1, ζ_2, ζ_3 can be obtained at once, that is, the position of the plane in which the three bodies move, in its relation to the invariable plane.

In order to get the six remaining coordinates, the five following equations are to be used.

$$(33)\qquad \begin{cases} \xi_1 + \xi_2 + \xi_3 = 0, \\ \eta_1 + \eta_2 + \eta_3 = 0, \\ \xi_1^2 + \eta_1^2 = r_{23}^2 - \zeta_1^2 = r_{23}', \\ \xi_2^2 + \eta_2^2 = r_{31}^2 - \zeta_2^2 = r_{31}', \\ \xi_3^3 + \eta_3^2 = r_{12}^2 - \zeta_3^2 = r_{12}', \end{cases}$$

and only one more relation is needed.

Putting

$$\tan\varphi = \frac{\eta_1}{\xi_1},$$

we get

$$(34)\quad d\varphi = \frac{\xi_1 d\eta_1 - \eta_1 d\xi_1}{\xi_1^2 + \eta_1^2} = \frac{\sqrt{(\xi_1^2+\eta_1^2)(d\xi_1^2+d\eta_1^2)-(\xi_1 d\xi_1+\eta_1 d\eta_1)^2}}{\xi_1^2+\eta_1^2}$$

$$= \frac{\sqrt{r_{23}'^2\left[\left(\frac{d\xi_1}{dt}\right)^2+\left(\frac{d\eta_1}{dt}\right)^2\right]-\left(r_{23}'\frac{dr_{23}'}{dt}\right)^2}}{r_{23}'^2}dt$$

The right-hand member is, from the preceding, a known function of t, consequently, (34) gives, by direct integration,

$$(35)\qquad \varphi = \int^t \frac{\sqrt{r_{23}'\left[\left(\frac{d\zeta_1}{dt}\right)^2+\left(\frac{d\eta_1}{dt}\right)^2\right]-\left(r_{23}'\frac{dr_{23}'}{dt}\right)^2}}{r_{23}'^2}.\,dt+C.$$

This and the five equations (33) determine the six required coordinates.

It appears, then, that the last integral (35) of the problem of three bodies is obtained by a simple quadrature. The reason for this has been given by Jacobi in his celebrated paper on the principle of the last multiplier (*Vorlesungen über Dynamik*), where he showed that the problem of three bodies belongs among those cases where the last multiplier is directly applicable if the other integrals are known.

8. Special Cases of the Problem of Three Bodies.

While the reductions given in the preceding paragraphs are the extreme ones permitted by the general problem, it is possible in several special cases to go farther and reduce the problem to a lower order or even to solve it completely. These cases will now be discussed.

I. *In which the plane of the three bodies remains constantly parallel to itself.*

In this case the plane of the three bodies is constantly the invariable plane. The right hand members of (32), § 7, and its

two corresponding equations, are $= 0$ because they are in the ratios of the distances of the three points from the invariable plane. The left hand members, which must also vanish, depend only on the distances and their time derivatives, and consequently we get three new integrals of the reduced problem, between which, however, an identical relation exists. They are perfectly symmetrical, and may be written

$$(1)\quad \frac{\begin{vmatrix} \xi_2, & \xi_3, & \frac{d\xi_1}{dt} \\ \eta_2, & \eta_3, & \frac{d\eta_1}{dt} \\ \zeta_2, & \zeta_3, & \frac{d\zeta_1}{dt} \end{vmatrix}}{m_1} = \frac{\begin{vmatrix} \xi_3, & \xi_1, & \frac{d\xi_2}{dt} \\ \eta_3, & \eta_1, & \frac{d\eta_2}{dt} \\ \zeta_3, & \zeta_1, & \frac{d\zeta_2}{dt} \end{vmatrix}}{m_2} = \frac{\begin{vmatrix} \xi_1, & \xi_2, & \frac{d\xi_3}{dt} \\ \eta_1, & \eta_2, & \frac{d\eta_3}{dt} \\ \zeta_1, & \zeta_2, & \frac{d\zeta_3}{dt} \end{vmatrix}}{m_3}.$$

To get the distances themselves, these determinants must be squared, and when squared by compounding the columns, give

$$(2)\quad \frac{\begin{vmatrix} [2,2], & [2,3], & [2,1'] \\ [2,3], & [3,3], & [3,1'] \\ [2,1'], & [3,1'], & [1',1'] \end{vmatrix}}{m_1^2} = \frac{\begin{vmatrix} [3,3], & [3,1], & [3,2'] \\ [2,1], & [1,1], & [1,2'] \\ [3,2'], & [1,2'], & [2',2'] \end{vmatrix}}{m_2^2} = \frac{\begin{vmatrix} [1,1], & [1,2], & [1,3'] \\ [1,2], & [2,2], & [2,3'] \\ [1,3'], & [2,3'], & [3',3'] \end{vmatrix}}{m_3^2}.$$

We shall soon see that the expressions in equations (1) and (2) are not alone equal, but also $= 0$.

For $[\lambda, \mu']$ and $[\lambda', \mu']$, their values from (31) and (33), § 7, and for ρ its value from (20) are to be used in order to get two new integrals of the reduced problem. By their help the order is reduced from seven to five, and finally, by the elimination of t, to four.

This result may also be reached directly. By taking the plane of the three bodies as the xy plane, $z = 0$, and hence the relative motions of the bodies may be expressed by a system of the eighth order, composed of four equations each of the second order. The reduced problem is less by unity, hence of the seventh order, because the direction of the coordinate axes plays no part in it. The equation of the kinetic energy and of the law of areas afford two integrals, which reduce it to the fifth and finally, by the elimination of t, it is reduced to the fourth order.

II. *In which the three areal integrals are equal to zero.*

Here $a = b = c = 0$, and the right hand members of (32), § 7, vanish identically, hence also the left hand members. Equations (1) and (2) remain as in the previous case. This leads to the conjecture that, in this case also, the plane passing through the three bodies remains constantly parallel to itself. This is capable of proof, as follows:

Developing the first determinant of system (1) by columns, we get it

$$= A\frac{d\xi_1}{dt} + B\frac{d\eta_1}{dt} + C\frac{d\zeta_1}{dt}, \tag{3}$$

in which

$$A = \eta_2\zeta_3 - \eta_3\zeta_2, \quad B = \zeta_2\xi_3 - \zeta_3\xi_2, \quad C = \xi_2\eta_3 - \xi_3\eta_2. \tag{4}$$

By equations (12), § 7, these expressions are unchanged when the subscripts 1, 2, 3 are cyclically interchanged.

The other determinants therefore become

$$\left\{\begin{aligned} &= A\frac{d\xi_2}{dt} + B\frac{d\eta_2}{dt} + C\frac{d\zeta_2}{dt}, \text{ and} \\ &= A\frac{d\xi_3}{dt} + B\frac{d\eta_3}{dt} + C\frac{d\zeta_3}{dt}. \end{aligned}\right. \tag{5}$$

Calling the three determinants of (1) p_1, p_2, p_3, we get

$$\frac{p_1}{m_1} = \frac{p_2}{m_2} = \frac{p_3}{m_3},$$

and further, identically,

$$p_1 + p_2 + p_3 = 0.$$

Designating the common value of the preceding fractions by λ,

$$p_1 = \lambda m_1, \quad p_2 = \lambda m_2, \quad p_3 = \lambda m_3,$$

hence

$$\lambda(m_1 + m_2 + m_3) = 0,$$

and as $m_1 + m_2 + m_3$ cannot vanish,

$$\lambda = 0, \quad \text{and hence} \quad p_1 = p_2 = p_3 = 0. \tag{6}$$

The quantities A, B, C, have a very simple geometrical meaning; they equal twice the three projections of the triangle whose

angles are at the three points. By their ratios, they therefore determine the position of the plane of these points. Equations (6) show that the relative motion of the points is in this plane, and the above proposition follows from it.

This case is, consequently, included in case I and affords no new reductions.

III. *In which the three areal integrals and the constant of the kinetic energy are equal to zero.*

In this case the system of the fifth order, (found after the elimination of t), can be reduced to an order lower by unity. This follows from a principle of great simplicity and importance, which may be viewed as an extension of Kepler's third law to the general problem of n bodies.

The differential equations of this problem in their original form are

$$* \quad \frac{d^2 x_i}{dt^2} = \frac{\partial V}{\partial x_i}.$$

The second members of these equations are homogeneous functions of the coordinates, of the degree — 2. If we write kx_i, ky_i, kz_i for x_i, y_i, z_i and $tk^{\frac{3}{2}}$ for t, k^{-2} will appear as a factor iu both members, and when it is suppressed, the equations have their original forms. From this it follows that from any solution x_i, y_i, z_i, another solution kx_i, ky_i, kz_i, with a constant factor k can be obtained, if $tk^{\frac{3}{2}}$ be substituted for t in the expressions for these coordinates as functions of t. This can be expressed as a proposition.

From any solution of the problem of **n** *bodies another can be deduced in which all the linear magnitudes are increased or decreased in a constant ratio. For such systems the squares of the times, from the first to the last places respectively, must be proportional to the cubes of the linear magnitudes.*

When attention is turned from the position of the orbits in space and time, and fixed only on their positions with reference to each other, the system can be reduced to the eighth order, as has been shown, without the introduction of a single constant of integration. If again the linear dimensions are not considered and only the form of the system is taken into account,

another arbitrary constant (the k above) is eliminated and the order is reduced by unity. But the two given integrals (that of the kinetic energy and that of the sum of the squares of the areas) determine but one of the two integrals in the second case. If the two constants of integration are given, they must, after the above substitution, be multiplied by determined powers of k (the first power + or —), and the product of the two is the only general integral for this case.

It is otherwise when these constants each = 0, for they do not then change on this substitution, but remain = 0. In this special case the last system of the seventh order is reduced by two units.

So long as we confine ourselves to real quantities, this case can occur only when the three areal integrals are separately zero. Then there is a further reduction of two units, and hence,

When the three areal integrals and the constant of the kinetic energy vanish, the given system can be reduced to the third order.

I will not delay farther on this case, as it is a special one for which the mathematical formulas can be at once written out after the above explanations.

IV. *In which the three distances of the three gravitating points remain constant.*

It must first be shown that this is possible.

For a given instant, assume that r_{12}, r_{23}, r_{31} are given and that their first and second time derivatives vanish. Then, for this instant, the quantities Q and the three derivatives of (25), § 7, also vanish, and we have from these equations, if the third derivatives of the time also vanish for the same instant,

$$\rho\left(\frac{1}{r_{12}^3} - \frac{1}{r_{13}^3}\right) = 0, \quad \rho\left(\frac{1}{r_{23}^3} - \frac{1}{r_{21}^3}\right) = 0, \quad \rho\left(\frac{1}{r_{31}^3} - \frac{1}{r_{32}^3}\right) = 0,$$

therefore either

$$\text{(a)} \quad r_{12}^3 = r_{23}^3 = r_{31}^3,$$

and, since they are real,

$$\text{(a)} \quad r_{12} = r_{23} = r_{31},$$

or

$$(b) \quad \rho = 0.$$

These two cases are to be considered separately.

$$(a) \quad r_{12} = r_{23} = r_{31}.$$

The triangle formed by joining the three points is, for any given instant, equilateral and the first and second time derivatives of the distances vanish. Under this assumption, derivatives higher than the third also vanish, that is the distances remain constant. For differentiating the first of equations (25), we get

$$\tfrac{1}{2}[1,1]'''' = \frac{dQ_1}{dt} + m_1\frac{d\rho}{dt}\left(\frac{1}{r_{12}^{\,3}} - \frac{1}{r_{13}^{\,3}}\right) - m_1\rho\left(\frac{3\dfrac{dr_{12}}{dt}}{r_{12}^{\,4}} - \frac{3\dfrac{dr_{13}}{dt}}{r_{13}^{\,4}}\right)$$

in which the right hand member vanishes for the given instant since $\dfrac{dQ_1}{dt}$ contains no term free from the first or second derivatives. In the same way it can be shown that derivatives of higher orders also vanish.

If r is the common value of the three distances, then by (26),

$$P_1 = P_2 = P_3 = -\frac{M}{r},$$

and (18) becomes

$$\begin{vmatrix} r^2, & -\frac{1}{2}r^2, & 0, & -\frac{\rho}{2} \\ -\frac{1}{2}r^2, & r^2, & +\frac{\rho}{2}, & 0 \\ 0, & +\frac{\rho}{2}, & \frac{M}{r}, & -\frac{1}{2}\frac{M}{r} \\ -\frac{\rho}{2}, & 0, & -\frac{1}{2}\frac{M}{r}, & \frac{M}{r} \end{vmatrix} = 0,$$

or
$$\left(\frac{\rho}{2}\right)^4 - \frac{3}{2}\left(\frac{\rho}{2}\right)^2 Mr + \frac{9}{16}M^2r^2 = 0,$$

$$\left(\frac{\rho}{2}\right)^2 = \tfrac{3}{4}Mr,$$

$$\rho^2 = 3Mr.$$

The first of the determinants in (2), this section, becomes

$$\begin{vmatrix} r^2, & -\frac{1}{2}r^2, & -\frac{\rho}{2} \\ -\frac{1}{2}r^2, & r^2, & +\frac{\rho}{2} \\ -\frac{\rho}{2}, & +\frac{\rho}{2}, & -\frac{M}{r} \end{vmatrix} = \left(\tfrac{3}{4}Mr - \frac{\rho^2}{4}\right)r^2.$$

and the two others likewise vanish, and this case becomes a special form of Case II. Equation (34) gives for the angular velocity with which the triangle turns in its plane

$$\frac{d\varphi}{dt} = \frac{\sqrt{r^2\frac{M}{r}}}{r^2} = \sqrt{\frac{M}{r^3}}.$$

If all the motions are referred to the center of gravity, it appears that they are as if the invariable triangle turned in its plane around its center of gravity with a velocity $\sqrt{\frac{M}{r^3}}$.

Or inversely, if the three gravitating points form the vertices of an equilateral triangle and if each has a velocity, in the direction perpendicular to the line joining it to the center of gravity, such that in the time dt, the angle through which this line turns $= \sqrt{\frac{M}{r^3}}\,dt$, then the triangle rotates as if the system were rigid.

(b) $\rho = 0$.

In this case, by (24)

$$\text{(I)}\quad [2,3]\,m_1\left(\frac{1}{r_{31}^3} - \frac{1}{r_{21}^3}\right) + [3,1]\,m_2\left(\frac{1}{r_{12}^3} - \frac{1}{r_{32}^3}\right) + [1,2]\,m_3\left(\frac{1}{r_{23}^3} - \frac{1}{r_{13}^3}\right) = 0,$$

and with the help of this equation, it appears that the distances are again constant if their first and second time derivatives vanish for a given instant.

From (18), it follows that

$$0 = \begin{vmatrix} [1,1], & [2,1] \\ [1,2], & [2,2] \end{vmatrix} \cdot \begin{vmatrix} P_1, & -\frac{1}{2}(P_1 + P_2 - P_3) \\ -\frac{1}{2}(P_1 + P_2 - P_3), & P_2 \end{vmatrix},$$

which is satisfied by either determinant vanishing.

If the first determinant vanishes, we have the equation

$$\begin{vmatrix} [1,1], [2,1] \\ [1,2], [2,2] \end{vmatrix} = 0,$$

which, by (23), § 7, becomes

$$(r_{12}+r_{23}+r_{31})(r_{12}+r_{23}-r_{31})(r_{12}-r_{23}+r_{31})(-r_{12}+r_{23}+r_{31})=0.$$

As the distances are positive, one of the last three factors only can vanish, and the equation shows that the longest side of the triangle equals the sum of the other two. Selecting r_{23} as the longest side, we get

$$\text{(II)} \qquad r_{23} = r_{12} + r_{13},$$

and in the case of real orbits, the only ones belonging to the physical problem, the points must be in one straight line. From (I) and (II) the ratios of the distances can be obtained.

If we represent r_{23}, r_{13}, r_{12} respectively by x_1, x_2, x_3, then

$$\text{(III)} \qquad x_1 = x_2 + x_3,$$

and

$$\begin{aligned} [2,3] &= -\tfrac{1}{2}(x_2^2 + x_3^2 - x_1^2) = +\, x_2\, x_3, \\ [3,1] &= -\tfrac{1}{2}(x_3^2 + x_1^2 - x_2^2) = -\, x_1\, x_3, \\ [1,2] &= -\tfrac{1}{2}(x_1^2 + x_2^2 - x_3^2) = -\, x_1\, x_2, \end{aligned}$$

and (I) becomes

$$\text{(IV)} \quad m_1(x_3^3 - x_2^3)x_1^2 - m_2(x_1^3 - x_3^3)x_2^2 - m_3(x_2^3 - x_1^3)\,x_3^2 = 0,$$

that is an equation of the fifth degree, while in general (I) is of the eighth degree.

The conditions (2) of this section are also fulfilled, and the relative motion about the center of gravity, or about any one of the given points, consists of a rotation in a plane of the straight line in which they lie, here assumed rigid. The angular velocity is, by (34),

$$\frac{d\varphi}{dt} = \frac{\sqrt{[1', 1']}}{r_{23}} = \frac{\sqrt{-P_1}}{r_{23}},$$

from which it appears that

$$\frac{\sqrt{P_1}}{r_{23}} = \frac{\sqrt{P_2}}{r_{31}} = \frac{\sqrt{P_3}}{r_{12}},$$

equations which can be obtained from (I) and (II).

Taking now the second case above, the equation

$$(\mathrm{V})\qquad 0 = \begin{vmatrix} P_1, & -\tfrac{1}{2}(P_1 + P_2 - P_3) \\ -\tfrac{1}{2}(P_1 + P_2 - P_3), & P_2 \end{vmatrix}$$

can be written

$$\sqrt{P_1} + \sqrt{P_2} + \sqrt{P_3} = 0.$$

Equation (V) can, with the aid of (I), be brought into a very elegant form. The transformation, which is not an easy one, is most simply performed as follows:

If, for brevity,

$$A = u[1, 1] + v[1, 2] + w[1, 3],$$
$$B = u[2, 1] + v[2, 2] + w[2, 3],$$
$$u = \frac{1}{r_{23}^{\ 3}},\quad v = \frac{1}{r_{31}^{\ 3}},\quad w = \frac{1}{r_{12}^{\ 3}},$$

then

$$P_1 = m_1 A - Mu\,[1, 1],$$
$$P_2 = m_2 B - Mv\,[2, 2].$$

The expression $-\dfrac{P_1 + P_2 - P_3}{2}$ can, from (I), be expressed in two forms, namely,

$$-\tfrac{1}{2}(P_1 + P_2 - P_3) = m_1 B - Mu\,[1, 2],$$
$$-\tfrac{1}{2}(P_1 + P_2 - P_3) = m_2 A - Mv\,[1, 2],$$

and (V) becomes

$$0 = M^2uv\,([1, 1]\,[2, 2] - [1, 2]^2)$$
$$\quad - M[m_1v\,(A[2,2] - B\,[1,2]) + m_2u\,(B\,[1,1] - A\,[1,2])].$$

But

$$A[2,2] - B[1,2] = u([1,1][2,2] - [1,2]^2) + w([1,3][2,2] - [2,3][1,2])$$
$$= (u - w)\,([1,1]\,[2,2] - [1,2]^2),$$

$$B[1,1] - A[1,2] = v([1,1][2,2] - [1,2]^2) + w([2,3][1,1] - [1,3]\,[1,2])$$
$$= (v - w)\,([1,1]\,[2,2] - [1,2]^2).$$

Hence,

$$0 = M\,([1,1]\,[2,2] - [1,2]^2)\,(m_1\,v\,w + m_2\,w\,u + m_3\,u\,v),$$

or

$$0 = M\,([1,1]\,[2,2] - [1,2]^2)\,(m_1\,r_{23}^{\ 3} + m_2\,r_{31}^{\ 3} + m_3\,r_{12}^{\ 3}).$$

This equation divides into three. The first and last factors can not vanish, as long as only positive masses are considered. There remains only the middle one, and that brings us back to the previous case where the three points lie in a straight line.

If negative masses are admitted into the problem of three bodies, the first or third factors might $= 0$. Then the planes of the three orbits no longer constantly coincide, but they rotate about an axis which is generally inclined to the original plane. The rotative orbits of two of the gravitating points are circles about the third, but their planes do not pass through it.

Moreover, it is worthy of note that the case in which the sum of the masses vanishes represents a remarkable limiting case of the problem of three bodies. There is then no center of gravity, the expressions Σmx, Σmy, Σmz depend only on the relative coordinates, and the six integrals of the center of gravity relate only to the relative motion. If there are only two bodies, the path of the one relative to the other is a straight line described with uniform velocity, and the relative coordinates are linear functions of the time. To get the actual path, these functions are to be inserted in the original differential equations (1), § 1, when they become determinable by direct quadrature. A special case is that in which the relative coordinates are constant, when each point describes a parabola in space.

If there are three bodies for which $m_1 + m_2 + m_3 = 0$, the order of the reduced problem, in general the seventh after use of the areal integrals and the proposition of kinetic energy, can be reduced by two, because equation (27), § 7, is at once integrable and gives

$$m_2 m_3 r_{23}^2 + m_3 m_1 r_{31}^2 + m_1 m_2 r_{12}^2 = Ct^2 + C_1 t + C_2.$$

But this can not be farther reduced by elimination of the time, for this is contained explicitly in the preceding expression.

V. *In which the angles of the triangle, formed by the three bodies, remain constant.*

In this case put

$$r_{23} = ur_1, \quad r_{31} = ur_2, \quad r_{12} = ur_3,$$

in which r_1, r_2, r_3 are constant and u alone is variable. Introducing these quantities into (25), § 7, we get

$$r_1^2 \frac{d^2\left(u\frac{du}{dt}\right)}{dt^2} = P_1' \frac{u\frac{du}{dt}}{u^3} + \rho \frac{A_1}{u^3}.$$

Here P_1' is what P in (26) becomes when r_1, r_2, r_3 are introduced for r_{23}, r_{31}, r_{12} and

$$\text{(VI)} \qquad A_1 = \left(\frac{1}{r_3^3} - \frac{1}{r_2^3}\right) m_1 .$$

In ρ remains yet u and its first and second derivatives in a somewhat complicated form, as is shown by (20), § 7. To get an equation in u, which shall contain not ρ but only $\frac{d\rho}{dt}$, the above equation must be differentiated again, and we get

$$\text{(VII)} \qquad r_1^2 \frac{d^3\left(\frac{u\,du}{dt}\right)}{dt^3} = P_1' \frac{d\left(\frac{du}{u^2\,dt}\right)}{dt} - 3\rho \frac{A_1}{u^4}\frac{du}{dt} + \frac{A_1}{u^3}\frac{d\rho}{dt}.$$

Multiplying the first equation by $3\frac{du}{u\,dt}$ and adding, it appears

$$\text{(VIIa)} \qquad r_1^2 \left(\frac{d^3\left(\frac{u\,du}{dt}\right)}{dt^3} + \frac{3}{u}\frac{du}{dt}\frac{d^2\left(u\frac{du}{dt}\right)}{dt^2}\right)$$
$$= P_1' \left(d\frac{\left(\frac{du}{u^2\,dt}\right)}{dt} + 3\frac{u\frac{du}{dt}}{u^3}\frac{du}{u\,dt}\right) + \frac{A_1}{u^4}\rho' ,$$

in which ρ' is the value of $\frac{d\rho}{dt}$ from (24), § 7, when r_1, r_2, r_3 are inserted in the place of r_{23}, r_{31}, r_{12}.

From (VII) can be formed two similar differential equations with the coefficients r_2^2, P_2', $A_2\rho'$ and r_3^3, P_3', $A_3\rho'$ respectively. If $\frac{du}{dt}$ does not $= 0$, that is, if the distances are not constant, then, as easily appears, must the coefficients of the equations be in proportion, and

$$\text{(VIII)} \qquad r_1^2 : r_2^2 : r_3^2 = P_1' : P_2' : P_3' = A_1\rho' : A_2\rho' : A_3\rho'.$$

These make four mutually consistent equations between the three quantities r_1, r_2, r_3.

Taking the first and third sets of ratios and assuming that ρ' does not vanish, or, if it vanishes, nevertheless,

$$r_1^2 : r_2^2 : r_3^2 = A_1 : A_2 : A_3,$$

it follows that

$$r_1^2 : r_2^2 : r_3^2 : \frac{r_1^2}{m_1} + \frac{r_2^2}{m_2} + \frac{r_3^2}{m_3} = A_1 : A_2 : A_3 : \frac{A_1}{m_1} + \frac{A_2}{m_2} + \frac{A_3}{m_3}.$$

But, by (VI),

$$\frac{A_1}{m_1} + \frac{A_2}{m_2} + \frac{A_3}{m_3} = 0.$$

Since $\frac{r_1^2}{m_1} + \frac{r_2^2}{m_2} + \frac{r_2^2}{m_3}$ cannot $= 0$, it follows that

$$A_1 = A_2 = A_3 = 0, \text{ and}$$

(IX) $$r_1 = r_2 = r_3.$$

We will assume that this $= 1$, then

$$r_{12} = r_{23} = r_{31} = u,$$

and the triangle is again equilateral.

The remaining conditions of (VIII) are likewise fulfilled, since it follows from (IX) that

$$P_1' = P_2' = P_3' = -M.$$

Equation (VII) now takes the form

(X) $$d^2 \frac{u \frac{du}{dt}}{dt^2} = -M \frac{u \frac{du}{dt}}{u^3}.$$

It is easy to show that, in this case also, the motion is in one plane. The differential equations (11), § 7, become now

$$\frac{d^2\xi_1}{dt^2} = -\frac{M\xi_1}{(\sqrt{\xi_1^2 + \eta_1^2 + \zeta_1^2})^3},$$

$$\frac{d^2\eta_1}{dt^2} = -\frac{M\eta_1}{(\sqrt{\xi_1^2 + \eta_1^2 + \zeta_1^2})^3},$$

$$\frac{d^2\zeta_1}{dt^2} = -\frac{M\zeta_1}{(\sqrt{\xi_1^2 + \eta_1^2 + \zeta_1^2})^3},$$

which are the differential equations for the relative motions of two gravitating points the sum of whose masses $= M$. Hence

The relative motion of any two bodies about a third is exactly the same as if the masses were united in the third, leaving the mass of the other two $= 0$.

There remains the case when $\rho = 0$, but not $A_1 = A_2 = A_3 = 0$. We then get from (VIII)

$$\begin{aligned} r_1^2 P_2' - r_2^2 P_1' &= 0, \\ r_2^2 P_3' - r_3^2 P_2' &= 0, \\ r_3^2 P_1' - r_1^2 P_3' &= 0. \end{aligned}$$

If ρ' is written out as it is found directly from (24), § 7, we get

$$\begin{aligned} r_1^2 P_2' - r_2^2 P_1' - \rho' [1,2] \\ = m_2 &\left(\frac{[2,2][1,1]-[1,2]^2}{r_2^3} + \frac{[3,2][1,1]-[1,2][1,3]}{r_3^3}\right) \\ - m_1 &\left(\frac{[2,2][1,1]-[1,2]^2}{r_1^3} + \frac{[3,1][2,2]-[2,3][1,3]}{r_3^3}\right) \\ - M &\left([2,2][1,1]-[1,2]^2\right)\left(\frac{1}{r_2^3} - \frac{1}{r_1^3}\right). \end{aligned}$$

But,

$$\begin{aligned} [2,2][1,1]-[1,2]^2 &= -([3,2][1,1]-[1,2][1,3]) \\ &= -([3,1][2,2]-[2,3][1,3]) \\ &= \triangle, \end{aligned}$$

when $\triangle$ is an abbreviation, as follows

$$\triangle = (r_1 + r_2 + r_3)(r_1 + r_2 - r_3)(r_1 - r_2 + r_3)(-r_1 + r_2 + r_3).$$

The preceding equation, since $\rho' = 0$, then becomes

$$\begin{aligned} r_1^2 P_2' - r_2^2 P_1' = \triangle \Big[m_1 \left(\frac{1}{r_3^3} - \frac{1}{r_2^3}\right) + m_2 \left(\frac{1}{r_1^3} - \frac{1}{r_3^3}\right) \\ - m_3 \left(\frac{1}{r_2^3} - \frac{1}{r_1^3}\right)\Big] = - \triangle (A_1 + A_2 - A_3). \end{aligned}$$

In the same manner

$$\begin{aligned} r_2^2 P_3' - r_3^2 P_2' &= - \triangle (A_1 - A_2 + A_3), \\ r_3^2 P_1' - r_1^2 P_3' &= - \triangle (- A_1 + A_2 + A_3), \end{aligned}$$

or, if it is not true that $A_1 = A_2 = A_3$, then $\triangle = 0$, that is, *the three points must be in one straight line.*

It appears that in this case also the orbits must be in a fixed plane and that the paths of two points about a third must be as if they were attracted by the third only. But the attraction of the third is not that of the sum of the masses, but a somewhat more complicated relation exists.

VI. *In which one mass, m_1, or two masses, m_1 and m_2, $= 0$.*

Let $m_1 = 0$, then the first of (25), § 7, passes over into

$$\frac{d^2\left(r_{23}\frac{dr_{23}}{dt}\right)}{dt^2} = -(m_2 + m_3)\frac{r_{23}\left(\frac{dr_{23}}{dt}\right)}{r_{23}^{\ 3}},$$

which can be at once integrated, so that we get r_{23} as a function of t. After substitution of r_{23} in the two other equations of (25), they become differential equations of the third order between r_{12}, r_{13} and t, of which there is no known integral except when, as Jacobi has shown, r_{23} is constant.

VI is nearly the case in the solar system for the three bodies, sun, earth and moon. The relative motion of sun and earth, or better of sun and the center of gravity of earth and moon, is nearly independent of the moon's position, and the latter is determined after the former. It is historically of interest that this idea was the first incentive to Lagrange's investigations.

If m_1 and $m_2 = 0$, r_{13} and r_{23}, and generally the relative motions of the first and second points about the third, may be directly determined by § 1. The motions of the first relative to the second then follow at once, and it appears that the third equation of (25), § 7, can also be integrated. This case is approximately that of several bodies with very small masses with reference to their relative motions about a single body with large mass, and these are closely the conditions for the sun and planets.

9. Historical Notes on the Problem of Three Bodies.

The investigations of the preceding paragraphs are especially due to Lagrange, who, in 1772, published his classical treatise entitled, *Essai d'une nouvelle méthode pour résoudre le problème des trois corps.* His most noteworthy advance was

in the formation of equation (34) for the determination of ρ; this was especially creditable to him because the theory of determinants was then in its infancy. Beyond this point he turned his investigation in the direction of the system formed by the sun, earth and moon, expressed in the symmetry which was always so sought by him. His greatest contribution to the problem was in its reduction to the seventh order,—a reduction which is equivalent to the production of a new integral.

Jacobi gave another kind of reduction in his memoir published in 1843, and entitled, *Sur l'élimination des noeuds dans le problème des trois corps* (*Crelle's Journal*, p. 115). He introduced into the differential equations the relative coordinates about the center of gravity and then considered two new fictitious gravitating points such that the coordinates of the three points, relative to the center of gravity, become linear functions of the coordinates of these fictitious points, and the conditions $\Sigma mx = \Sigma my = \Sigma mz = 0$ are identically fulfilled. When these new coordinates are introduced into the expressions for the potential and the kinetic energy, the latter take on a somewhat different form. The system thus formed is of the twelfth order, but is reduced by four by the aid of the areal integrals, and the law of the kinetic energy, and it then appears that, by making the invariable plane the plane of reference and introducing polar coordinates, the problem can be so transformed that the nodes of the two variable planes are determinable by a simple quadrature and there remains a system of the seventh order. This method is remarkably elegant, yet its talented author did not succeed in reducing the problem beyond the point reached by Lagrange.

The reduction by one order in the differential equations, accomplished by Lagrange and Jacobi in such different ways, can be performed on the equations in their original form, as has been shown by Radau, *Sur l'élimination directe du noeud dans le problème des trois corps*, (*Comptes rendus*, LXVII, 1868 p., 841) and by Allegret in his *Mémoire sur le problème des trois corps* (*Journal de Mathématiques*, 1875, p. 277).

The labors of later mathematicians have resulted in many

noteworthy transformations of the problem of three bodies. Bertrand and Bour (*Mémoire sur le problème des trois corps, Journal de l'école polytechnique*, 1856, p. 35) have brought the reduced problem of the eighth order, by ten known integrals, to a canonical form in which the function H does not explicitly contain the time. This will receive attention in the following sections. Bour has succeeded in putting the expression for kinetic energy into such a form that it separates into two distinct parts. One part depends only on the form and position of the triangle made by the bodies, and on its position and changes of position with reference to the invariable plane. By using this form Mathieu (*Mémoire sur le problème des trois corps, Journal de Mathématiques*, 1876, p. 345), by introducing a system of eight variables, which have a very simple geometrical meaning, has succeeded in obtaining a canonical group of differential equations.

However interesting these studies may be, they do not reduce the order of the final system. This has been done, so far, only in the two cases where either the motions are in one plane, or all three areal integrals = 0. The second case is, as has been shown, only a special case of the first (only for three bodies, however), because the motion must be in one plane when the three areal integrals = 0. The order can then be reduced by two, as shown by Allegret in the memoir cited above. Allegret thought that by special transformations this simplification could be made general, but he fell into an error which Mathieu has pointed out. Another simplification which was introduced by Hesse in his memoir, *Ueber das Problem der drei Körper*, (*Crelle's Journal*, 1887, p. 47), also proves illusory, as J. A. Serret has shown, in a note to Lagrange's memoir. This celebrated problem still stands where it was left by Lagrange, a century ago, and though a multitude of versions of the problem have been developed, Bruns shows (*Acta Mathematica* XI, p. 43), that they give no prospect of its more complete algebraic solution.

SECOND DIVISION.

The General Properties of the Integrals.

10. Poisson's and Lagrange's Formulas.

While it is true that only the integrals already given have been so far obtained, yet they do not include every result of value in the problem of n bodies. Certain highly important properties have been discovered, both of the known integrals and of the unknown. While these properties were at first deduced by Poisson and Lagrange from the system which has been discussed, they have been shown by Jacobi to be fundamental and to belong to a much more general system. His profound investigations, the framework of which will be given in what follows, have been generalized and simplified by Professor A. Mayer and Professor Sophus Lie. To arrive at them, it is convenient to bring into another form the differential equations of § 6.

If the components of the velocity of the bodies be introduced as new independent variables by means of the equations

$$(1)\quad \frac{dx_i}{dt}=u_i,\qquad \frac{dy_i}{dt}=v_i,\qquad \frac{dz_i}{dt}=w_i,\qquad (i=1,2\ldots n),$$

the equations of motion become

$$(2)\qquad m_i\frac{du_i}{dt}=\frac{\partial V}{\partial x_i},\quad m_i\frac{dv_i}{dt}=\frac{\partial V}{dy_i},\quad m_i\frac{dw_i}{dt}=\frac{\partial V}{\partial z_i},$$

when, as before,

$$(3)\quad V=\sum\frac{m_\lambda m_\mu}{r_{\lambda\mu}}=\sum\frac{m_\lambda\, m_\mu}{\sqrt{(x_\lambda-x_\mu)^2+(y_\lambda-y_\mu)^2+(z_\lambda-z_\mu)^2}}.$$

By the introduction of u_i, v_i, w_i as new variables, their number is doubled, and also that of the differential equations (1) and (2) form a closed system of simultaneous differential

equations. At the same time, all the differential equations are now of the first order.

Equations (1) can be put in the same form as equations (2), when, for the kinetic energy, we put

(4) $$T = \tfrac{1}{2}\Sigma m_i\,(u_i^2 + v_i^2 + w_i^2).$$

For then

$$\frac{\partial T}{\partial u_i} = m_i u_i,$$

and (1) become

$$m_i \frac{dx_i}{dt} = \frac{\partial T}{\partial u_i}, \quad m_i \frac{dy_i}{dt} = \frac{\partial T}{\partial v_i}, \quad m_i \frac{dz_i}{dt} = \frac{\partial T}{\partial w_i}.$$

This can be still farther simplified. For, if in the place of u_i, v_i, w_i, other new variables be introduced by the equations

(5) $$U_i = m_i u_i, \quad V_i = m_i v_i, \quad W_i = m_i w_i,$$

we get

(6) $$T = \frac{1}{2}\sum \frac{1}{m_i}\,(U_i^2 + V_i^2 + W_i^2),$$

$$\frac{\partial T}{\partial u_i} = m_i u_i = m_i \frac{\partial T}{\partial U_i}.$$

Finally, putting

(7) $$H = -V + T = -\sum \frac{m_\lambda m_\mu}{r_{\lambda,\mu}} + \frac{1}{2}\sum \frac{1}{m_i}(U_i^2 + V_i^2 + W_i^2),$$

and noting x_i, y_i, z_i occur only in V, and U_i, V_i, W_i only in T, we get the equations of motion in the following form

(8) $$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial U_i}, & \dfrac{dy_i}{dt} = \dfrac{\partial H}{\partial V_i}, \quad \dfrac{dz_i}{dt} = \dfrac{\partial H}{\partial W_i}, \\ \dfrac{dU_i}{dt} = -\dfrac{\partial H}{\partial x_i}, & \dfrac{\partial V_i}{dt} = -\dfrac{\partial H}{\partial y_i}, \quad \dfrac{\partial W_i}{dt} = -\dfrac{\partial H}{\partial z_i}. \end{cases}$$

For the same of simplicity employ p_i for x_i, y_i, z_i, and q_i for U_i, V_i, W_i, and n for $3n$, and the equations (8) become

(9) $$\begin{cases} \dfrac{dp_i}{dt} = \dfrac{\partial H}{\partial q_i}, \\ \dfrac{dq_i}{dt} = -\dfrac{\partial H}{\partial p_i} \end{cases} \qquad (i = 1, 2, \ldots n).$$

To pass from (9) to (8), we have only to substitute $3n$ for n; x_i, y_i, z_i for p_i; U_i, V_i, W_i for q_i.

The differential equations of motion have thus been brought to the form (9). This form can be given to very many problems of mechanics, and since, when once found, they recur after the most various transformations, and from them are most easily developed the dependent formulas, they are said to be in *cononical form.*

In order to give all possible generality, we will assume that *the function H of (9) is any given function of p_i and q_i, and also of the time t.* With this entirely general assumption for H, the equations (9) can be treated as follows:

Multiply equations (9) by $\frac{\partial H}{\partial p_i}$ and $\frac{\partial H}{\partial q_i}$, and we get, by addition,

$$\sum\left(\frac{\partial H}{\partial p_i}\frac{dp_i}{dt}+\frac{\partial H}{dq_i}\frac{dq_i}{dt}\right)=0. \tag{10}$$

If H does not contain t, this passes at once into

$$\frac{dH}{dt}=0, \tag{11}$$

and hence

$$H = C, \tag{12}$$

when C is the constant of integration. If H contains t, equation (10) becomes

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}, \tag{13}$$

a non-integrable equation.

The integration of equations (9) introduces $2n$ arbitrary constants, $a_1, a_2, a_3, \ldots a_{2n}$. To Lagrange is due the conception, that these constants may be regarded as independent analytical magnitudes, an idea of the greatest importance in very different ways. The object of the integration is to obtain p_i and q_i as functions of t and of the $2n$ constants. The resulting expressions must be of the form

$$\begin{cases} p_i = p_i\,(a_1, a_2, a_3 \ldots a_{2n}, t) \\ q_i = q_i\,(a_1, a_2, a_3 \ldots a_{2n}, t) \end{cases} \quad (i = 1, 2 \ldots n), \tag{14}$$

where, in the second members, the parentheses contain the functions which are to determine the quantities placed before the parentheses. (This notation is commendable for its clearness and simplicity.)

If equations (14) are solved for the a's, the results are functions of p_i, q_i, t, of the form

$$(15) \qquad a_i = a_i(p_1, \ldots p_n, q_1, \ldots q_n, t).$$

Equations (14) are the inverse of equations (15), and conversely. For equations (14) may be obtained from equations (15) in the same way that the latter were obtained from the former, t merely playing the part of a parameter.

It is evident that, in the choice of constants a_i, there is much arbitrariness. For if a system $a_1, \ldots a_{2n}$ has been selected, $2n$ independent, constant functions of them may be introduced into (14) and (15), and thus we may pass from the given system to a new one, $a_1', a_2', \ldots a_{2n}'$. This is a circumstance which will later prove of great usefulness.

As a result of this, we can consider the integration of the canonical equations as dependent on the $2n$ equations (15); that is, we can form $2n$ functions of p_i, q_i, t, which, with the help of (9), have the property of becoming constants. Jacobi, in this narrower sense, calls such functions integrals of (9). *The integrals are therefore functions of p_i, q_i, t, which contain no arbitrary constants, but which with the assistance of* (9) *become themselves arbitrary constants.*

These functions satisfy a linear partial differential equation of the first order, which may be regarded as their defining equation. For if $a(p_1, p_2, \ldots p_n, q_1, q_2, \ldots q_n, t)$ is such a function, then

$$(15a) \qquad \frac{da}{dt} = 0,$$

or by total differentiation,

$$(16) \qquad 0 = \frac{\partial a}{\partial t} + \sum \frac{\partial a}{\partial p_i} \frac{\partial p_i}{dt} + \sum \frac{\partial a}{dq_i} \frac{dq_i}{dt} = 0.$$

This equation by (9) is transformed into

$$(17)\qquad 0=\frac{\partial a}{\partial t}+\sum\frac{\partial a}{\partial p_i}\frac{\partial H}{\partial q_i}-\sum\frac{\partial a}{\partial q_i}\frac{\partial H}{\partial p_i},$$

an equation which must be identical, since it contains no arbitrary constant. It is the partial differential equation mentioned above. Conversely, if the function a satisfies the equation (17), we can obtain (16) and ($15a$) again from this; therefore a is an integral.

The discovery of 2n integrals of (9) *coincides with the determination of 2n particular solutions of* (17), *and thus the problem is again reduced to the consideration of this partial differential equation.* The identical integral, $a=$ constant, is not here considered.

We shall now introduce a much simpler method. If f and φ are two functions of p_i and q_i, (which may also contain t), we can put

$$(18)\qquad (f,\varphi)=\sum\frac{\partial f}{\partial p_i}\frac{\partial\varphi}{\partial q_i}-\sum\frac{\partial f}{\partial q_i}\frac{\partial\varphi}{\partial p_i},$$

so that, $(f,\varphi)=-(\varphi,f)$, and $(f,f)=0$.

The differential equation (17) then takes the form

$$(19)\qquad \frac{\partial a}{\partial t}+(a,H)=0,$$

And from this we can proceed, in the following manner, to Poisson's Theorem:

The three functions f, φ, ψ are functions of p_i and q_i, and they may also contain t. Hence

$$\Big((f,\varphi),\psi\Big)=\sum_{i'=1}^{i'=n}\left(\frac{\partial(f,\varphi)}{\partial p_{i'}}\cdot\frac{\partial\psi}{\partial q_{i'}}-\frac{\partial(f,\varphi)}{\partial q_{i'}}\cdot\frac{\partial\psi}{\partial p_{i'}}\right)$$

in which another summation index i' has been selected. Substituting the value of (f,φ) from (18), gives

$$\Big((f,\varphi),\psi\Big)=\sum_{i=1}^{i=n}\sum_{i'=1}^{i'=n}\frac{\partial^2 f}{\partial p_i\,\partial p_{i'}}\cdot\frac{\partial\varphi}{\partial q_i}\cdot\frac{\partial\psi}{\partial q_{i'}}+\sum_{i=1}^{i=n}\sum_{i'=1}^{i'=n}\frac{\partial^2\varphi}{\partial q_i\,\partial p_{i'}}\cdot\frac{\partial f}{\partial p_i}\cdot\frac{\partial\psi}{\partial q_{i'}}$$

$$-\sum_{i=1}^{i=n}\sum_{i=1}^{i'=n}\frac{\partial^2 f}{\partial q_i\,\partial p_{i'}}\cdot\frac{\partial\varphi}{\partial p_i}\cdot\frac{\partial\psi}{\partial q_{i'}}-\sum_{i=1}^{i=n}\sum_{i'=1}^{i'=n}\frac{\partial^2\varphi}{\partial p_i\,\partial p_{i'}}\cdot\frac{\partial f}{\partial q_i}\cdot\frac{\partial\psi}{\partial q_{i'}}$$

$$-\sum_{i=1}^{i=n}\sum_{i'=1}^{i'=n}\frac{\partial^2 f}{\partial p_i \partial q_{i'}}\cdot\frac{\partial \varphi}{\partial q_i}\cdot\frac{\partial \psi}{\partial p_{i'}}-\sum_{i=1}^{i=n}\sum_{i'=1}^{i'=n}\frac{\partial^2 \varphi}{\partial q_i \partial q_{i'}}\cdot\frac{\partial f}{\partial p_i}\cdot\frac{\partial \psi}{\partial p_{i'}}$$

$$+\sum_{i=1}^{i=n}\sum_{i'=1}^{i'=n}\frac{\partial^2 f}{\partial q_i \partial q_{i'}}\cdot\frac{\partial \varphi}{\partial p_i}\cdot\frac{\partial \psi}{\partial p_{i'}}+\sum_{i=1}^{i=n}\sum_{i'=1}^{i'=n}\frac{\partial^2 \varphi}{\partial p_i \partial q_{i'}}\cdot\frac{\partial f}{\partial q_i}\cdot\frac{\partial \psi}{\partial p_{i'}}.$$

The four double summations containing the second derivatives of f are collectively symmetrical with respect to φ and ψ, as may be seen by permuting the summation indices. Their algebraic sum, for an instant, being symbolized by

$$\Big(f,[\varphi,\psi]\Big)=\Big(f,[\psi,\varphi]\Big),$$

the above equation becomes

$$\Big((f,\varphi,),\psi\Big)=\Big(f,[\varphi,\psi]\Big)-\Big(\varphi,[f,\psi]\Big).$$

Adding to this the similar expressions for $\Big((\varphi,\psi),f\Big)$ and $\Big((\psi,f),\varphi\Big)$, gives the identical equation,

$$(20)\qquad \Big((f,\varphi),\psi\Big)+\Big((\varphi,\psi),f\Big)+\Big((\psi,f),\varphi\Big)=0,$$

by means of which Poisson's formula may be easily obtained.

Differentiating (19) with respect to $p_{i'}$ and $q_{i'}$, which may be done by assuming that it becomes identical on the substitution of a from (15), we get

$$(21)\qquad \begin{cases} \dfrac{\partial \frac{\partial a}{\partial p_{i'}}}{\partial t}+\dfrac{\partial (a,H)}{\partial p_i}=0, \\[2ex] \dfrac{\partial \frac{\partial a}{\partial q_{i'}}}{\partial t}+\dfrac{\partial (a,H)}{\partial q_{i'}}=0. \end{cases}$$

Multiplying the first of these equations by $\dfrac{\partial b}{\partial q_{i'}}$ and the second by $\dfrac{\partial b}{\partial p_{i'}}$, where b denotes any function of p and q, and taking the sum with respect to i', gives

$$(22)\qquad \sum_{i'=1}^{i'=n}\left(\frac{\partial b}{\partial q_{i'}}\cdot\frac{\partial \frac{\partial a}{\partial p_{i'}}}{\partial t}-\frac{\partial b}{\partial p_{i'}}\cdot\frac{\partial \frac{\partial a}{\partial q_{i'}}}{\partial t}\right)+\Big((a,H),b\Big)=0.$$

Exchanging a and b on the assumption that b is also an integral, and subtracting the two equations gives

$$(23) \sum_{i=1}^{i'=n} \left(\frac{\partial \frac{\partial a}{\partial p_{i'}}}{\partial t} \cdot \frac{\partial b}{\partial q_{i'}} - \frac{\partial \frac{\partial a}{\partial q_{i'}}}{\partial t} \cdot \frac{\partial b}{\partial p_{i'}} - \frac{\partial a}{\partial q_{i'}} \cdot \frac{\partial \frac{\partial b}{\partial p_{i'}}}{\partial t} + \frac{\partial a}{\partial p_{i'}} \cdot \frac{\partial \frac{\partial b}{\partial q_{i'}}}{\partial t} \right)$$
$$+ \Big((a, H), b \Big) - \Big((b, H), a \Big) = 0.$$

The first term is $= \frac{\partial (a, b)}{\partial t}$. Further

$$\Big((b, H), a \Big) = \Big(-(H, b), a \Big) = - \Big((H, b), a \Big),$$

and consequently equation (23), by the use of (20), becomes

$$(24) \qquad \frac{\partial (a, b)}{\partial t} + \Big((a, b), H \Big) = 0.$$

But this equation may be obtained from (19) by writing (a, b) for a. This establishes Poisson's proposition:

(25) *If a and b are two integrals of the system* (9), *then* (a, b) *is also an integral of this system.*

By applying this proposition, we see that

If a and b are two integrals of the system (8), *then*

$$(26) \ (a, b) = \sum_{i=1}^{i=n} \frac{1}{m_i} \left(\frac{\partial a}{\partial x_i} \cdot \frac{\partial b}{\partial u_i} - \frac{\partial a}{\partial u_i} \cdot \frac{\partial b}{\partial x_i} + \frac{\partial a}{\partial y_i} \cdot \frac{\partial b}{\partial v_i} \right.$$
$$\left. - \frac{\partial a}{\partial v_i} \cdot \frac{\partial b}{\partial y_i} + \frac{\partial a}{\partial z_i} \cdot \frac{\partial b}{\partial w_i} - \frac{\partial a}{\partial w_i} \cdot \frac{\partial b}{\partial z_i} \right)$$

is also an integral of the same system.

Hence it follows that, by pure differentiation, Poisson's proposition furnishes a new integral from two given integrals. The integral thus obtained may be entirely new, or it may be a function of the two already known, or it may even reduce to a constant or zero. If it is a new integral, it may be combined with the original ones, giving two more integrals, and this process may be continued until the integrals obtained are merely combinations of the earlier ones. This will occur when $2n$ integrals have been obtained, or the cycle may close earlier. The following investigations illustrate these interesting conditions.

As an application of Poisson's proposition, we will obtain the third areal integral from the other two. We have

$$a = \Sigma m_i (y_i w_i - z_i v_i),$$
$$b = \Sigma m_i (z_i u_i - x_i w_i).$$

And, according to (26), these give

$$(a, b) = \sum \frac{1}{m_i} (m_i 0 \,.\, z_i + m_i 0 \,.\, w_i + m_i w_i \,.\, 0 - m_i z_i \,.\, 0 + m_i v_i \,.\, m_i x_i$$
$$- m_i y_i \,.\, m_i u_i) = \Sigma m_i (x_i v_i - y_i u_i),$$

which is the third areal integral. With this the cycle is closed. The application of Poisson's proposition to all the previous integrals furnishes no new integrals.

The areal integrals, the integrals relating to the motion of the center of gravity and the integrals relating to the kinetic energy, each form a closed system of integrals.

We shall next consider Lagrange's proposition. After substituting equations (14) in (9), the latter must be identically satisfied for each value of $a_1, a_2, \ldots a_{2n}, t$. Hence, equations (9) may be identically differentiated with respect to a_λ. This gives

$$\frac{\partial^2 p_i}{\partial a_\lambda \partial t} = \frac{\partial \dfrac{\partial H}{\partial q_i}}{\partial a_\lambda}$$
$$= \sum_{i'=1}^{i'=n} \left(\frac{\partial^2 H}{\partial q_i \partial p_{i'}} \cdot \frac{\partial p_i}{\partial a_\lambda} + \frac{\partial^2 H}{\partial q_i \partial q_{i'}} \cdot \frac{\partial q_{i'}}{\partial a_\lambda} \right),$$

and similarly,

$$\frac{\partial^2 q_i}{\partial a_\lambda \partial t} = - \sum_{i'=1}^{i'=n} \left(\frac{\partial^2 H}{\partial p_i \partial p_{i'}} \cdot \frac{\partial p_{i'}}{\partial a_\lambda} + \frac{\partial^2 H}{\partial p_i \partial q_{i'}} \cdot \frac{\partial q_{i'}}{\partial a_\lambda} \right).$$

Now let a_μ denote a new constant, and multiply the first equation by $\dfrac{\partial q_i}{\partial a_\mu}$ and the second by $- \dfrac{\partial p_i}{\partial a_\mu}$ and add the products. The sum with respect to i is

$$\sum_{i=1}^{i=n} \left(\frac{\partial^2 p_i}{\partial a_\lambda d t} \cdot \frac{\partial q_i}{\partial a_\mu} - \frac{\partial^2 q_i}{\partial a_\lambda \partial t} \cdot \frac{\partial p_i}{\partial a_\mu} \right) =$$

$$\sum_{i=1}^{i=n}\sum_{i'=1}^{i'=n}\left(\frac{\partial^2 H}{\partial q_i \partial p_{i'}}\cdot\frac{\partial p_{i'}}{\partial a_\lambda}\cdot\frac{\partial q_i}{\partial a_\mu}+\frac{\partial^2 H}{\partial q_i \partial q_{i'}}\cdot\frac{\partial q_{i'}}{\partial a_\lambda}\cdot\frac{\partial q_i}{\partial a_\mu}\right.$$
$$\left.+\frac{\partial^2 H}{\partial p_i \partial p_{i'}}\cdot\frac{\partial p_{i'}}{\partial a_\lambda}\cdot\frac{\partial p_i}{\partial a_\mu}+\frac{\partial^2 H}{\partial p_i \partial q_{i'}}\cdot\frac{\partial q_{i'}}{\partial a_\lambda}\cdot\frac{\partial p_i}{\partial a_\mu}\right).$$

The second member is symmetrical with respect to a_λ and a_μ, consequently

$$\sum_{i=1}^{i=n}\left(\frac{\partial^2 p_i}{\partial a_\lambda\, \partial t}\cdot\frac{\partial q_i}{\partial a_\mu}-\frac{\partial^2 q_i}{\partial a_\lambda\, \partial t}\cdot\frac{\partial p_i}{\partial a_\mu}-\frac{\partial^2 p_i}{\partial a_\mu\, \partial t}\cdot\frac{\partial q_i}{\partial a_\lambda}\right.$$
$$\left.+\frac{\partial^2 q_i}{\partial a_\mu\, \partial t}\cdot\frac{\partial p_i}{\partial a_\lambda}\right)=0,$$

or

$$(27)\qquad \frac{d}{dt}\left[\sum_{i=1}^{i=n}\left(\frac{\partial p_i}{\partial a_\lambda}\cdot\frac{\partial q_i}{\partial a_\mu}-\frac{\partial q_i}{\partial a_\lambda}\cdot\frac{\partial p_i}{\partial a_\mu}\right)\right]=0.$$

This equation proves Lagrange's proposition; viz:

The expression

$$(28)\qquad [a_\mu, a_\lambda]=\sum_{i=1}^{i=n}\left(\frac{\partial p_i}{\partial a_\lambda}\cdot\frac{\partial q_i}{\partial a_\mu}-\frac{\partial q_i}{\partial a_\lambda}\cdot\frac{\partial p_i}{\partial a_\mu}\right)$$

being independent of t, is only a function of the constants a.

Moreover the expressions $[a_\lambda, a_\mu]$ have also the property, that

$$(29)\qquad [a_\mu, a_\lambda]=-[a_\lambda, a_\mu] \text{ and } [a_\lambda, a_\lambda]=0.$$

For the system (8)

$$(30)\quad [a_\lambda, a_\mu]=\sum_{i=1}^{i=n} m_i\left(\frac{\partial x_i}{\partial a_\lambda}\cdot\frac{\partial u_i}{\partial a_\mu}-\frac{\partial u_i}{\partial a_\lambda}\cdot\frac{\partial x_i}{\partial a_\mu}+\frac{\partial y_i}{\partial a_\lambda}\cdot\frac{\partial v_i}{\partial a_\mu}\right.$$
$$\left.-\frac{\partial v_i}{\partial a_\lambda}\cdot\frac{\partial y_i}{\partial a_\mu}+\frac{\partial z_i}{\partial a_\lambda}\cdot\frac{\partial w_i}{\partial a_\mu}-\frac{\partial w_i}{\partial a_\lambda}\cdot\frac{\partial z_i}{\partial a_\mu}\right).$$

Lagrange's formula will not, like Poisson's, furnish new results from certain conditions, for the formation of the expressions $[a_\lambda, a_\mu]$ already requires a knowledge of all the equations (14) and hence of the solution of the problem. Nevertheless the two formulas are so closely related that each may be regarded

as the converse of the other. Lagrange, although he mentions Poisson's formula, seems to have overlooked this relationship. In order to determine this relationship, we shall collect some proportions in determinants for use in what follows.

Let it be assumed that the determinant

$$(31) \qquad \Delta = \begin{vmatrix} a_{1,1}, & a_{1,2}, & \dots & a_{1,2n} \\ a_{2,1}, & a_{2,2}, & \dots & a_{1,2n} \\ \dots & . & \dots & \dots \\ a_{2n,1}, & a_{2n,2}, & \dots & a_{2n,2n} \end{vmatrix}$$

of the $2n^{th}$ order is not equal to zero. The minor belonging to any element $a_{\lambda,\mu}$ will, after having been divided by Δ, be denoted $A_{\mu,\lambda}$, so that

$$(32) \qquad A_{\mu,\lambda} = \frac{1}{\Delta} \frac{\partial \Delta}{\partial a_{\lambda,\mu}} \, .$$

Likewise let a determinant be formed of the A's,

$$(33) \qquad D = \begin{vmatrix} A_{1,1}, & A_{1,2}, & \dots & A_{1,2n} \\ A_{2,1}, & A_{2,2}, & \dots & A_{2,2n} \\ \dots & \dots & \dots & \dots \\ A_{2n,1}, & A_{2n,2}, & \dots & A_{2n,2n} \end{vmatrix}$$

On account of the conjugate relation between Δ and D, we shall call these conjugate determinants. From the second determinant

$$(34) \qquad a_{\mu,\lambda} = \frac{1}{D} \cdot \frac{\partial D}{\partial A_{\lambda,\mu}}$$

and at the same time

$$(35) \qquad \Delta D = 1 \, .$$

From the known relations among the elements of Δ and D, we have

$$(36) \qquad a_{\lambda,1} A_{1,\lambda} + a_{\lambda,2} A_{2,\lambda} + \dots + a_{\lambda,2n} A_{2n,\lambda} = 1,$$

$$(37) \qquad a_{\lambda,1} A_{1,\mu} + a_{\lambda,2} A_{2,\mu} + \dots + a_{\lambda,2n} A_{2n,\mu} = 0, \; \mu \gtrless \lambda,$$

and likewise

$$(38) \qquad a_{1,\lambda} A_{\lambda,1} + a_{2,\lambda} A_{\lambda,2} + \dots + a_{2n,\lambda} A_{\lambda,2n} = 1,$$

$$(39) \qquad a_{1,\lambda} A_{\mu,1} + a_{2,\lambda} A_{\mu,2} + \dots + a_{2n,\lambda} A_{\mu,2n} = 0, \; \mu \gtrless \lambda \, .$$

If we form the $2n$ equations

(40) $\quad a_{1,\lambda} x_1 + a_{2,\lambda} x_2 + \ldots + a_{2n,\lambda} x_{2n} = y_\lambda, \; (\lambda = 1, \ldots 2n),$

and solve them with reference to $x_1, x_2, \ldots x_{2n}$ as unknown quantities, the results are of the form

(41) $$A_{1,\lambda} y_1 + A_{2,\lambda} y_2 + \ldots + a_{2n,\lambda} y_{2n} = x_\lambda$$

and, in the same manner, the solution of the $2n$ equations

(42) $\quad a_{\lambda,1} x_1 + a_{\lambda,2} x_2 + \ldots + a_{\lambda,2n} x_{2n} = y_\lambda, \; (\lambda = 1, \ldots 2n),$

gives

(43) $$A_{\lambda,1} y_1 + A_{\lambda,2} y_2 + \ldots + A_{\lambda,2n} y_{2n} = x_\lambda.$$

These are the most important properties of the conjugate determinants $\varDelta$ and D, and they are also valid when the determinants are of an odd order. The developments which follow are valid for determinants of an even order only.

If the square of $\varDelta$ is formed by combining the rows or columns according to the rule for multiplication of determinants, the result is a symmetrical determinant. By a slight modification, as—so far as I know—was first shown by Brioschi, this may be made a skew symmetrical determinant, that is, one in which the constituents of the principal diagonal are all zeros and those symetrically placed with respect to this diagonal, are equal but with opposite signs. This may be shown by taking the 1^{st} row of (31) with the negative of the $(n+1)^{th}$ row, the 2^{nd} row with the negative of the $(n+2)^{th}$ row, and so on. In order to make this plainer, replace the a's in the last n rows by b's, so that the determinant becomes

(44) $$\varDelta = \begin{vmatrix} a_{1,1}, a_{1,2}, \ldots & a_{1,2n} \\ a_{2,1}, a_{2,2}, \ldots & a_{2,2n} \\ \ldots \quad \ldots \quad \ldots & \ldots \\ a_{n,1}, a_{n,2}, \ldots & a_{n,2n} \\ b_{1,1}, b_{1,2}, \ldots & b_{1,2n} \\ \ldots \quad \ldots \quad \ldots & \ldots \\ b_{n,1}, b_{n,2}, \ldots & b_{n,2n} \end{vmatrix}.$$

The transformed determinant then becomes

$$(45) \qquad \Delta' = \begin{vmatrix} -b_{1,1}, -b_{1,2}, \ldots & -b_{1,2n} \\ \ldots \quad \ldots \quad \ldots & \ldots \\ -b_{n,1}, -b_{n,2}, \ldots & -b_{n,2n} \\ +a_{1,1}, +a_{1,2}, \ldots & +a_{1,2n} \\ \ldots \quad \ldots \quad \ldots & \ldots \\ +a_{n,1}, +a_{n,2}, \ldots & +a_{n,2n} \end{vmatrix}.$$

Further, if the columns are compounded, it becomes

$$(46) \quad E = \Delta\,\Delta' = \Delta^2 = \begin{vmatrix} c_{1,1}, c_{1,2}, \ldots & c_{1,2n} \\ c_{2,1}, c_{2,2}, \ldots & c_{2,2n} \\ \ldots \quad \ldots \quad \ldots & \ldots \\ c_{2n,1}, c_{2n,2}, \ldots & c_{2n,2n} \end{vmatrix}$$

where, in general,

$$(47) \qquad \begin{aligned} c_{\lambda,\mu} = {} & a_{1,\lambda} b_{1,\mu} + a_{2,\lambda} b_{2,\mu} + \ldots + a_{n,\lambda} b_{n,\mu} \\ & - b_{1,\lambda} a_{1,\mu} - b_{2,\lambda} a_{2,\mu} - \ldots - b_{n,\lambda} a_{n,\mu}, \end{aligned}$$

and hence, as should be the case

$$c_{\lambda,\mu} + c_{\mu,\lambda} = 0.$$

Further, denoting the constituents of the last n columns of (33) by the letters B, it becomes

$$(48) \quad D = \begin{vmatrix} A_{1,1}, A_{1,2}, \ldots & A_{1,n}, B_{1,2}, \ldots & B_{1,n} \\ \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots & \ldots \\ \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots & \ldots \\ A_{2n,1}, A_{2n,2}, \ldots & A_{2n,n}, B_{2n,1}, \ldots & B_{2n,n} \end{vmatrix}.$$

By writing D in the new form

$$(49) \quad D' = \begin{vmatrix} -B_{1,1}, -B_{1,2}, \ldots & +A_{1,1}, \ldots & +A_{1,n} \\ \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots & \ldots \\ \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots & \ldots \\ -B_{2n,1}, -B_{2n,2}, \ldots & +A_{2n,1}, \ldots & +A_{2n,n} \end{vmatrix}$$

and combining its horizontal rows, it becomes

$$(50) \quad F = DD' = D^2 = \begin{vmatrix} C_{1,1}, & C_{1,2}, \ldots & C_{1,2n} \\ \ldots & \ldots \quad \ldots & \ldots \\ \ldots & \ldots \quad \ldots & \ldots \\ C_{2n,1} & C_{2n,2}, \ldots & C_{2n,2n} \end{vmatrix}$$

where

(51)
$$C_{\mu,\lambda} = A_{\lambda,1} B_{\mu,1} + \ldots + A_{\lambda,n} B_{\mu,n} - B_{\lambda,1} A_{\mu,1} - \ldots - B_{\lambda,n} A_{\mu,n},$$

and also

$$C_{\lambda,\mu} + C_{\mu,\lambda} = 0.$$

It follows at once that

$$E F = \varDelta^2 D^2 = 1.$$

But a still closer relation exists between the skew determinants E and F. Like (44) and (48), or $\varDelta$ and D, they are conjugate. For, in accordance with the definition of conjugate determinants

(52)
$$A_{\lambda,\mu} = \frac{1}{\varDelta} \frac{\partial \varDelta}{\partial a_{\mu,\lambda}}, \qquad B_{\lambda,\mu} = \frac{1}{\varDelta} \frac{\partial \varDelta}{\partial b_{\mu,\lambda}},$$

(53)
$$a_{\lambda,\mu} = \frac{1}{D} \frac{\partial D}{\partial A_{\mu,\lambda}}, \qquad b_{\lambda,\mu} = \frac{1}{D} \frac{\partial D}{\partial B_{\mu,\lambda}}.$$

In order to derive the corresponding expressions for E and F, we write (47), with the aid of (53), in the form

$$c_{\lambda,\mu} = \frac{1}{D^2}\Big(\frac{\partial D}{\partial A_{\lambda,1}} \frac{\partial D}{\partial B_{\mu,1}} + \ldots + \frac{\partial D}{\partial A_{\lambda,n}} \frac{\partial D}{\partial B_{\mu,n}} - \frac{\partial D}{\partial B_{\lambda,1}} \frac{\partial D}{\partial A_{\mu,1}} - \ldots - \frac{\partial D}{\partial B_{\lambda,n}} \frac{\partial D}{\partial A_{\mu,n}} \Big),$$

or, also with regard to (49),

$$c_{\lambda,\mu} = \frac{1}{D D'}\Big(\frac{\partial D}{\partial B_{\mu,1}} \frac{\partial D'}{\partial A_{\lambda,1}} + \ldots + \frac{\partial D}{\partial B_{\mu,n}} \frac{\partial D'}{\partial A_{\lambda,n}} + \frac{\partial D}{\partial A_{\mu,1}} \frac{\partial D'}{\partial (-B_{\lambda,1})} + \ldots + \frac{\partial D}{\partial A_{\mu,n}} \frac{\partial D'}{\partial (-B_{\lambda,\mu})} \Big).$$

Consequently, $c_{\lambda,\mu}$ is equal to $\frac{1}{D D'}$ times the sum of the products of the minors of the μ^{th} row of (48) with the corresponding minors of the λ^{th} row of (49). According to an extension of the theorem for the multiplication of determinants, $c_{\lambda,\mu} = \frac{1}{DD'}$ times the minors of the product of (48) and (49) and therefore of F which contains the term $C_{\lambda,\mu}$.

Therefore

(54)
$$c_{\lambda,\mu} = \frac{1}{F} \frac{\partial F}{\partial C_{\mu,\lambda}},$$

and likewise

$$(55)\qquad C_{\lambda,\mu}=\frac{1}{E}\frac{\partial E}{\partial c_{\mu,\lambda}},$$

and consequently E and F are conjugate determinants.

Therefore, if $x_1, \ldots, x_{2n}$ is a system of $2n$ variables and if a second system $y_1, \ldots, y_{2n}$ is determined by the linear substitutions

$$(56)\quad c_{1,\lambda}x_1+c_{2,\lambda}x_2+\ldots+c_{2n,\lambda}x_{2n}=y_\lambda \quad (\lambda=1,2,\ldots,2n),$$

then

$$(57)\quad C_{1,\lambda}y_1+C_{2,\lambda}y_2+\ldots+C_{2n,\lambda}y_{2n}=x_\lambda \quad (\lambda=1,2,\ldots,2n)$$

are the solutions of these equations.

Still other relations exist among the four determinants D, Δ, E, F. By allowing the index μ in (47) to take all values from 1 to $2n$ and solving the resulting $2n$ equations for $a_{1,\lambda}, \ldots, a_{n,\lambda}$, $b_{1,\lambda}, \ldots, b_{n,\lambda}$, we obtain

$$(58)\quad \begin{cases} \;\;\, a_{\mu,\lambda}=c_{\lambda,1}B_{1,\mu}+c_{\lambda,2}B_{2,\mu}+\ldots+c_{\lambda,2n}B_{2n,\mu}, \\ -b_{\mu,\lambda}=c_{\lambda,1}A_{1,\mu}+c_{\lambda,2}A_{2,\mu}+\ldots+c_{\lambda,2n}A_{2n,\mu}. \end{cases}$$

These equations show that the determinant Δ' may be obtained identically by combining the rows of E with the columns of D. By proceeding in the same manner with (51), we get

$$(59)\quad \begin{cases} \;\;\, A_{\lambda,\mu}=C_{1,\lambda}b_{\mu,1}+\ldots+C_{2n,\lambda}b_{\mu,2n}, \\ -B_{\lambda,\mu}=C_{1,\lambda}a_{\mu,1}+\ldots+C_{2n,\lambda}a_{\mu,2n}. \end{cases}$$

These developments of the determinants can be directly applied to Poisson's and to Lagrange's Theorems. The equations (15) are the solutions of (14). If they are substituted in (14), the equations become identical, and in this sense the differentiation of (14), therefore, gives

$$(60)\quad \begin{cases} 0=\dfrac{\partial p_\lambda}{\partial a_1}\dfrac{\partial a_1}{\partial p_\mu}+\ldots+\dfrac{\partial p_\lambda}{\partial a_{2n}}\dfrac{\partial a_{2n}}{\partial p_\mu}+\ldots \;(\lambda \lessgtr \mu), \\ 0=\dfrac{\partial p_\lambda}{\partial a_1}\dfrac{\partial a_1}{\partial q_\mu}+\ldots+\dfrac{\partial p_\lambda}{\partial a_{2n}}\dfrac{\partial p_{2n}}{\partial q_\mu}+\ldots \\ 1=\dfrac{\partial p_\lambda}{\partial a_1}\dfrac{\partial p_1}{\partial p_\lambda}+\ldots+\dfrac{\partial p_\lambda}{\partial a_{2n}}\dfrac{\partial a_{2n}}{\partial p_\lambda}+\ldots \end{cases}$$

and so on. And likewise, by the substitution of (14) in (15)

$$
(61) \qquad \begin{cases} 0 = \dfrac{\partial a_\lambda}{\partial p_1}\dfrac{\partial p_1}{\partial a_\mu} + \ldots + \dfrac{\partial a_\lambda}{\partial q_1}\dfrac{\partial q_1}{\partial a_\mu} + \ldots & (\lambda \lessgtr \mu) \\ 1 = \dfrac{\partial a_\lambda}{\partial p_1}\dfrac{\partial p_1}{\partial a_\lambda} + \ldots + \dfrac{\partial a_\lambda}{\partial q_1}\dfrac{\partial q_1}{\partial a_\lambda} + \ldots \end{cases}
$$

Therefore, if we put,

$$
(62) \qquad \begin{cases} \dfrac{\partial a_\lambda}{\partial p_\mu} = a_{\mu,\lambda}, \\ \dfrac{\partial a_\lambda}{\partial q_\mu} = b_{\mu,\lambda}, \\ \dfrac{\partial p_\mu}{\partial a_\lambda} = A_{\lambda,\mu}, \\ \dfrac{\partial q_\mu}{\partial a_\lambda} = B_{\lambda,\mu} \end{cases}
$$

in which $\lambda = 1, 2, \ldots 2n$ and $\mu = 1, 2, \ldots n$, the quantities $a_{\lambda,\mu}$, $b_{\lambda,\mu}$, $A_{\mu,\lambda}$, $B_{\mu,\lambda}$ form two conjugate systems D and Δ. The result of the combination here becomes

$$
c_{\lambda,\mu} = \frac{\partial a_\lambda}{\partial p_1}\frac{\partial a_\mu}{\partial q_1} + \frac{\partial a_\lambda}{\partial p_2}\frac{\partial a_\mu}{\partial q_2} + \ldots - \frac{\partial a_\lambda}{\partial q_1}\frac{\partial a_\mu}{\partial p_1} - \frac{\partial a_\lambda}{\partial q_2}\frac{\partial a_\mu}{\partial p_2} - \ldots,
$$

or simply,

$$
(63) \qquad c_{\lambda,\mu} = (a_\lambda, a_\mu).
$$

and, further,

$$
C_{\mu,\lambda} = \frac{\partial p_1}{\partial a_\lambda}\frac{\partial q_1}{\partial a_\mu} + \frac{\partial p_2}{\partial a_\lambda}\frac{\partial q_2}{\partial a_\mu} + \ldots - \frac{\partial p_1}{\partial a_\mu}\frac{\partial q_1}{\partial a_\lambda} - \frac{\partial p_2}{\partial a_\mu}\frac{\partial q_2}{\partial a_\lambda} - \ldots,
$$

or simply,

$$
(64) \qquad C_{\mu,\lambda} = [a_\lambda, a_\mu].
$$

Therefore, it appears, that if Poisson's n^2 expressions and Lagrange's n^2 expressions are arranged in determinants, the two determinants are conjugate and in this consists the remarkable relation between the two theorems.

11. Development of Poisson's and Lagrange's Formulas for the Elliptic Elements of the Orbit of a Planet.

The formulas of the preceding paragraphs can here be applied at once. If we put

(1) $$\frac{dx}{dt}=u, \quad \frac{dy}{dt}=v, \quad \frac{dz}{dt}=w,$$

then

(2) $$H=\frac{u^2+v^2+w^2}{2}-\frac{\mu}{\sqrt{x^2+y^2+u^2}},$$

and the differential equations of the motion become

(3) $$\begin{cases} \dfrac{dx}{dt}=\dfrac{\partial H}{\partial u}, & \dfrac{dy}{dt}=\dfrac{\partial H}{\partial v}, & \dfrac{dz}{dt}=\dfrac{\partial H}{\partial w}, \\ \dfrac{du}{dt}=-\dfrac{\partial H}{\partial x}, & \dfrac{dv}{dt}=-\dfrac{\partial H}{\partial y}, & \dfrac{dw}{dt}=-\dfrac{\partial H}{\partial z}, \end{cases}$$

In this case, equations (26) and (30) of § 10 become

(4) $$(a_\lambda, a_\mu)=\frac{\partial a_\lambda}{\partial x}\frac{\partial a_\mu}{\partial u}+\frac{\partial a_\lambda}{\partial y}\frac{\partial a_\mu}{\partial v}+\frac{\partial a_\lambda}{\partial z}\frac{\partial a_\mu}{\partial w} -\frac{\partial a_\lambda}{\partial u}\frac{\partial a_\mu}{\partial x}-\frac{\partial a_\lambda}{\partial v}\frac{\partial a_\mu}{\partial y}-\frac{\partial a_\lambda}{\partial w}\frac{\partial a_\mu}{\partial z},$$

(5) $$[a_\lambda, a_\mu]=\frac{\partial x}{\partial a_\lambda}\frac{\partial u}{\partial a_\mu}+\frac{\partial y}{\partial a_\lambda}\frac{\partial v}{\partial a_\mu}+\frac{\partial z}{\partial a_\lambda}\frac{\partial w}{\partial a_\mu} -\frac{\partial u}{\partial a_\lambda}\frac{\partial x}{\partial a_\mu}-\frac{\partial v}{\partial a_\lambda}\frac{\partial y}{\partial a_\mu}-\frac{\partial w}{\partial a_\lambda}\frac{\partial z}{\partial a_\mu}.$$

The six constants of integration are the elements of the orbit. In order, they are

6 $$\begin{cases} a_1=a= \text{the mean distance}, \\ a_2=e= \text{eccentricity}. \\ a_3=\Omega= \text{longitude of the ascending node}, \\ a_4=i= \text{inclination of the plane of the orbit}, \\ a_5=\pi= \text{longitude of perihelion}, \\ a_6=\varepsilon= \text{mean longitude of the planet for the time } t=0. \end{cases}$$

In this case it is best first to compute Lagrange's quantites $[a_\lambda, a_\mu]$. Then, by substituting for Ξ and H their values in terms of t, and remembering that Z is zero, the formulas (26a) § 1 can be at once used. The component velocities may be immediately obtained by writing $\frac{d\Xi}{dt}$ or Ξ' instead of Ξ and $\frac{dH}{dt}$ or H' instead of H, in equations (26 a).

Therefore, putting

(7) * $$x = \Xi\xi_x + H\eta_x$$

in which ζ_x, η_w, etc., denote the six coefficients of (26 a), we get

(8) * $$u = \Xi'\xi_x + H'\eta_x.$$

The derivatives with respect to Ω and i are the easiest to obtain since they alone appear in the ζ_x's and η_x's. Hence their determination presents no difficulty. The element a appears twice, explicitly as a factor, and implicitly in the angle

$$M = nt + \varepsilon - \pi = t\sqrt{\frac{\mu}{a^3}} + \varepsilon - \pi.$$

Denoting the entire derivatives by enclosing them within parentheses and leaving the derivatives, which arise from the explicit appearance of a, without them, we have

$$\left(\frac{\partial x}{\partial a}\right) = \frac{\partial x}{\partial a} + \frac{\partial x}{\partial M}\frac{\partial M}{\partial a} = \frac{\partial x}{\partial a} - \frac{3}{2}\frac{\partial x}{\partial \varepsilon}\frac{tn}{a} = \frac{x}{a} - \frac{3}{2}\frac{tu}{a},$$

etc.

Remembering that, in $u = \frac{\partial x}{\partial t} = \frac{\partial x}{\partial M}\sqrt{\frac{\mu}{a^3}}$, the element a appears only in the power $-\frac{1}{2}$, it likewise follows that

$$\left(\frac{\partial u}{\partial a}\right) = -\frac{u}{2a} - \frac{3}{2}\frac{t}{a}\frac{du}{dt} = -\frac{u}{2a} + \frac{3}{2}\frac{tx}{ar^3}, \text{ etc.}$$

The element ε appears only in $M = nt + \varepsilon - \pi$. Hence

(9) $$\begin{cases} \dfrac{\partial x}{\partial \varepsilon} = \dfrac{\partial x}{\partial t}\dfrac{1}{n} = \dfrac{u}{n}, \\[2ex] \dfrac{\partial u}{\partial \varepsilon} = \dfrac{\partial u}{\partial t}\dfrac{1}{n} = -\dfrac{\mu x}{nr^3}, \text{ etc.} \end{cases}$$

The element π appears twice, once in Ξ and H, and again explicitly in the ζ_x's and η_x's.

Denoting the entire derivatives with respect to π by enclosing them within parentheses and the parts of the same, which are obtained by the differentiation of ξ_x and η_x, without them, and remembering that π appears in M only in the relation $\varepsilon - \pi$, we have

$$(10)\qquad \begin{cases} \left(\frac{\partial x}{\partial \pi}\right)=\frac{\partial x}{\partial \pi}-\frac{\partial x}{\partial \varepsilon}, \\ \left(\frac{\partial u}{\partial \pi}\right)=\frac{\partial u}{\partial \pi}-\frac{\partial u}{\partial \varepsilon}. \end{cases}$$

It is somewhat more difficult to form the derivatives with respect to e. From the equations

$$(11)\qquad \begin{cases} \xi = a(\cos E - e) \\ \eta = a\sqrt{1-e^2}\sin E \end{cases}$$

we obtain,

$$(12)\qquad \begin{cases} \xi' = -an\dfrac{\sin E}{1-e\cos E} \\ \eta' = \quad an\dfrac{\sqrt{1-e^2}\cos E}{1-e\cos E} \end{cases}$$

In order to form the derivatives of ξ, η, ξ', η' with respect to e, it is only necessary to know $\frac{\partial E}{\partial e}$. By differentiating

$$M = E - e\sin E,$$

we get,

$$(13)\qquad \frac{\partial E}{\partial e}=\frac{\sin E}{1-e\cos E}.$$

With these preparations it is not difficult to compute the thirty-six derivatives of the coordinates and the component velocities with respect to the elements.

In order to form the expressions $[a_\lambda, a_\mu]$, the number of which in this case is essentially fifteen, a special value of t can be at once used in the thirty-six derivatives. We can, for example, put $M=E=0$ and thereby much reduce the necessary computations.

Moreover, five of the combinations can be immediately obtained in another way.

The equation for the kinetic energy, in this case, is

$$(14)\qquad H=-\frac{\mu}{2a},$$

which, by the substitution of the coordinates and velocities in terms of the time and elements, must become identical. In

this sense, the differentiation of equation (14) with respect to any element k, (a being excepted), gives

$$(15) \qquad \frac{\partial H}{\partial x}\frac{\partial x}{\partial k}+\ldots+\frac{\partial H}{\partial u}\frac{\partial u}{dk}+\ldots=0.$$

Using equations (3), and remembering that t appears only in $nt+\varepsilon$, we have

$$\frac{dx}{dt}=n\frac{\partial x}{\partial \varepsilon}+\ldots+\frac{\partial u}{\partial t}=n\frac{\partial u}{\partial \varepsilon}+\ \ldots,$$

and it follows immediately from (15), that

$$[\varepsilon, k]=0.$$

Hence

$$(16) \qquad [\varepsilon, e]=[\varepsilon, \Omega]=[\varepsilon, i]=[\varepsilon, \pi]=0.$$

But if k is the mean distance a, the second member of (15) is not 0, but $\frac{\mu}{2a^2}$, and

$$-n\,[a, \varepsilon]=\frac{\mu}{2a^2},$$

hence

$$(17) \qquad [a, \varepsilon]=-\tfrac{1}{2}\sqrt{\frac{\mu}{a}}.$$

The element ε is thereby fully exhausted. With the help of the above preparations, the ten remaining combinations can be obtained without difficulty. The results are as follows:

$$(18) \qquad \begin{cases} [a, \varepsilon]=-\tfrac{1}{2}\sqrt{\dfrac{\mu}{a}}, \\ [a, \Omega]=\dfrac{1-\cos i}{2}\sqrt{\dfrac{\mu(1-e^2)}{a}}, \\ [a, \pi]=\tfrac{1}{2}(1-\sqrt{1-e^2})\sqrt{\dfrac{\mu}{a}} \\ [e, \Omega]=-e(1-\cos i)\sqrt{\dfrac{\mu a}{1-e^2}}, \\ [e, \pi]=e\sqrt{\dfrac{\mu a}{1-e^2}}, \\ [\Omega, i]=-\sin i\sqrt{\mu a(1-e^2)}, \end{cases}$$

The remaining Lagrange's combinations are all $=0$. Consequently, the determinant F, (50) § 10, becomes

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} a_1=a & a_2=e & a_3=\Omega & a_4=i & a_5=\pi & a_6=\varepsilon \end{array} \\
\begin{array}{c} a \\ e \\ \Omega \\ i \\ \pi \\ \varepsilon \end{array} &
(19)\ F=\left|\begin{array}{cccccc}
0, & 0, & [a,\Omega], & 0, & [a,\pi], & [a,\varepsilon] \\
0, & 0, & [e,\Omega], & 0, & [e,\pi], & 0 \\
-[a,\Omega], & -[e,\Omega], & 0, & -[\Omega,i], & 0, & 0 \\
0, & 0, & -[\Omega,i], & 0, & 0, & 0 \\
-[a,\pi], & -[e,\pi], & 0, & 0, & 0, & 0 \\
-[a,\varepsilon], & 0, & 0, & 0, & 0, & 0
\end{array}\right|
\end{array}
$$

All the elements of this determinant to the right of the second diagonal are zeros, consequently it is equal to the negative product of the elements of this diagonal, or

$$(20)\qquad F=([a,\varepsilon][\Omega,i][e,\pi])^2=\tfrac{1}{4}\mu^3 a e^2 \sin^2 i.$$

Since so many of the elements are zeros, the calculation of the minor determinants is easy. Dividing them by F gives

$$
(21)\qquad \left\{\begin{aligned}
(a,\varepsilon) &= -2\sqrt{\frac{a}{\mu}}, \\
(e,\pi) &= \frac{1}{e}\sqrt{\frac{1-e^2}{\mu a}}, \\
(e,\varepsilon) &= +\frac{(1-\sqrt{1-e^2})\sqrt{1-e^2}}{e\sqrt{\mu a}}, \\
(\Omega,i) &= -\frac{1}{\sin i\sqrt{\mu a(1-e^2)}}, \\
(i,\pi) &= +\frac{1-\cos i}{\sin i\sqrt{\mu a(1-e^2)}}, \\
(i,\varepsilon) &= -\frac{1-\cos i}{\sin i\sqrt{\mu a(1-e^2)}},
\end{aligned}\right.
$$

All the remaining Poison's combinations are $=0$. The determinal E, (46) § 10, accordingly becomes

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} a & e & \Omega & i & \pi & \varepsilon \end{array} \\
\begin{array}{c} a \\ e \\ \Omega \\ i \\ \pi \\ \varepsilon \end{array} &
(22)\ E=\left|\begin{array}{cccccc}
0, & 0, & 0, & 0, & 0, & -(a,\varepsilon) \\
0, & 0, & 0, & 0, & -(e,\pi), & -(e,\varepsilon) \\
0, & 0, & 0, & -(\Omega,i), & 0, & 0 \\
0, & 0, & (\Omega,i), & 0, & -(i,\pi), & (i,\varepsilon) \\
0, & (e,\pi), & 0, & (i,\pi), & 0, & 0 \\
(a,\varepsilon), & (e,\varepsilon), & 0, & -(i,\varepsilon), & 0, & 0
\end{array}\right|
\end{array}
$$

These tables, (19) and (22), were given simultaneously by Lagrange and Laplace. They show that nine of the combinations $[a_\lambda, a_\mu]$ and (a_λ, a_μ) are equal to zero and that they do not contain the elements ε, π and Ω. By a proper selection of elements which are functions of these, it is possible to carry the reductions further. This will not be done, but in the following paragraphs a general theory will be developed which can easily be applied to these special cases.

12. The Canonical System of Constants of Integration.

The developments of §10 will now be continued with the use of the same notation.

If a and b are any two given functions of p_i and q_i, a third function of them is defined by Poisson's expression (a, b). Consequently, if a and b are two integrals, which may also contain the time, (a, b) represents a third integral. Again, if $a_1, a_2, \ldots a_{2n}$ is a complete system of integrals, there are, neglecting the sign, $\frac{1}{2}n(n-1)$ such combinations, and we shall show that they all take very simple values when a proper selection of the integrals is made.

In order to reach the fundamental idea of the investigation, we shall next consider Poisson's expressions (a, b) for themselves, without reference to their use in the theory of the integrals of our differential equations.

Let $a_1, a_2, \ldots a_m$ be any number of given functions of p_i and q_i, and, further, let b be a given function of $a_1, a_2, \ldots a_m$, which does not explicitely contain p_i and q_i.

Then

$$(a_\lambda, b) = \sum_{i=1}^{i=n}\left(\frac{\partial a_\lambda}{\partial p_i}\frac{\partial b}{\partial q_i} - \frac{\partial a_\lambda}{\partial q_i}\frac{\partial b}{\partial p_i}\right)$$

$$= \sum_{i=1}^{i=n}\left[\frac{\partial a_\lambda}{\partial p_i}\left(\frac{\partial b}{\partial a_1}\frac{\partial a_1}{\partial q_i} + \frac{\partial b}{\partial a_2}\frac{\partial a_2}{\partial q_i} + \ldots\right)\right.$$

$$\left. - \frac{\partial a_\lambda}{\partial q_i}\left(\frac{\partial b}{\partial a_1}\frac{\partial a_1}{\partial p_i} + \frac{\partial b}{\partial a_2}\frac{\partial a_2}{\partial p_i} + \ldots\right)\right],$$

Consequently,

$$(1)\quad (a_\lambda, b) = (a_\lambda, a_1)\frac{\partial b}{\partial a_1} + (a_\lambda, a_2)\frac{\partial b}{\partial a_2} + \ldots + (a_\lambda, a_m)\frac{\partial b}{\partial a_m}$$

$$= \sum_{\mu=1}^{\mu=m} (a_\lambda, a_\mu)\frac{\partial b}{\partial a_\mu},$$

Further, if b_1 is a function of a, satisfying the same conditions as a, it follows that

$$(2)\qquad (b, b_1) = \sum_{\lambda=1}^{\lambda=m}\sum_{\mu=1}^{\mu=m} (a_\lambda, a_\mu)\frac{\partial b}{\partial a_\lambda}\frac{\partial b_1}{\partial a_\mu}.$$

After these preparations, which show that Poisson's combination of "functions of functions" can be at once traced back to the first functions, we assume that α_1 is any given function of p_i and q_i. Then let another function β_1 of p_i and q_i be so determined that

$$(3)\qquad (\alpha_1, \beta_1) = 1.$$

Then, with respect to β_1, this is a linear partial differential equation of the first order, with p_i and q_i as independent variables; and β_1 may, therefore, be selected with all the arbitrariness allowed by the theory of these equations. Now let β_1 be determined in any fixed way.

The partial differential equation

$$(4)\qquad (\alpha_1, b) = 0$$

for a third function b has $(2n-1)$ independent integrals and the general integral is an arbitrary function of them. As one of these integrals, we may take α_1, since it gives the identity $(\alpha_1, \alpha_1) = 0$. Let $b_1, b_2, \ldots b_{2n-2}$ be the remaining $(2n-2)$ independent integrals of (4). If we substitute α_1, β_1, b for f, φ, ψ, in the fundamental relation (20), page 82, [§ 10], we immediately obtain, by reason of (3) and (4),

$$(5)\qquad (\alpha_1, (\beta_1, b)) = 0,$$

that is, (β_1, b) is also an integral of the partial differential equation (4) in b, and consequently the $(2n-2)$ expressions $(\beta_1, b_1), \ldots (\beta_1, b_{2n-2})$ depend entirely upon $\alpha_1, b_1, \ldots b_{2n-2}$. Let

c be any selected function of these $2n-1$ elements. Then, from (1),

$$(6) \qquad (\beta_1\, c) = -\frac{\partial c}{\partial a_1} + (\beta_1, b_1)\frac{\partial c}{\partial b_1} + \ldots + (\beta_1, b_{2n-2})\frac{\partial c}{\partial b_{2n-2}}.$$

Since the coefficients are functions of a_1 and $b_1, \ldots b_n$ only, it follows that

$$(7) \qquad (\beta_1, c) = 0.$$

This is a partial differential equation in c, with $a_1, b_1, \ldots b_{2n-2}$ as independent variables. It has $2n-2$ independent integrals, which may be denoted by $c_1, c_2, \ldots c_{2n-2}$. Since they are functions of $a_1, b_1, \ldots b_{2n-2}$, they also satisfy the equation

$$(8) \qquad (a_1, c) = 0.$$

The $(2n-2)$ functions $c_1, c_2, \ldots c_{2n-2}$ have now the property that any combination (c_λ, c_μ) depends on them alone. They form, according to Lie, a group. This may easily be shown as follows:

Since $a_1, \beta_1, c_1, \ldots c_{2n-2}$ are $2n$ functions of p_i and q_i, between which there is no identical relation, every other function of p_i and q_i, including (c_λ, c_μ), can be expressed by them. But since (c_λ, c_μ) satisfies the condition $(\beta_1, (c_\lambda, c_\mu)) = 0$, it cannot contain a_1 and since it satisfies the further condition $(a_1, (c_\lambda, c_\mu)) = 0$, it cannot contain β_1. Therefore, as stated above, (c_λ, c_μ) is only a function of c itself.

From the quantities c, select one c_1 and call it a_2. If β_2 is then a function of c, the equation

$$(9) \qquad (a_2, \beta_2) = (a_2, c_2)\frac{\partial \beta_2}{\partial c_2} + \ldots\ldots = 1$$

is a partial differential equation in β_2 with $a_2, c_2, \ldots c_{2n-2}$ as independent variables and β_2 is one of its integrals. From the differential equation

$$(10) \qquad (a_2, d) = (a_2, c_2)\frac{\partial d}{\partial c_2} + \ldots = 0,$$

which has $(2n-4)$ independent integrals besides $d = a_2$, we shall, on the contrary, select them all and denote them by $d_1, \ldots d_{2n-4}$. By reason of (9) and (10) it follows again from (20), § 10, that

(β_2, d) depend only upon $\alpha_2, d_1, \dots d_{2n-4}$. Letting e denote a function of $\alpha_2, d_1, \dots d_{2n-4}$,

$$(11) \quad (\beta_1, e) = -\frac{\partial e}{\partial \alpha_2} + (\beta_1, d_1)\frac{\partial e}{\partial d_1} + \dots + (\beta_1, d_{2n-4})\frac{\partial e}{\partial d_{2n-4}} = 0,$$

is a partial differential equation in e with $\alpha_2, d_1, \dots d_{2n-4}$ as independent variables, and has $(2n-4)$ independent integrals $e_1, \dots e_{2n-4}$. Since $\alpha_2, \beta_2, e_1 \dots e_{2n-4}$ are functions of $c_1, \dots c_{2n-4}$, they likewise satisfy the equations (7) and (8), and, consequently,

$$(\alpha_1, \alpha_2) = (\beta_1, \beta_2) = (\alpha_1, \beta_2) = (\alpha_2, \beta_1) = 0$$

and

$$(\alpha_1, e_\lambda) = (\beta_1, e_\lambda) = 0.$$

The functions e form a group, as did the functions c; therefore the combination (e_λ, e_μ) is a pure function of e. Now, consider this as a function of p_i and q_i, and restore the $2n$ functions $\alpha_1, \beta_1, \alpha_2, \beta_2, e_1, \dots e_{2n-4}$. Since

$$(\alpha_1, (e_\lambda, e_\mu)) = (\alpha_2, (e_\lambda, e_\mu)) = (\beta_1, (e_\lambda, e_\mu)) = (\beta_2, (e_\lambda, e_\mu)) = 0,$$

it appears that it does not contain $\alpha_1, \alpha_2, \beta_1, \beta_2$.

It is evident that this process which we have followed through two steps, can be continued indefinitely. New partial differential equations are successively obtained containing always a decreasing number of independent variables, and no restriction is made in the selection of the integrals. In the end, the following result is reached:

If α_1 be any given function of p_i and q_i, then, in an indefinite number of ways, $(2n-1)$ other functions $\alpha_2, \dots \alpha_n, \beta_1 \dots \beta_n$, may be determined, such that,

$$(12) \qquad \left\{ \begin{array}{l} (\alpha_\lambda, \beta_\lambda) = 1 \\ (\alpha_\lambda, \beta_\mu) = 0 \\ (\alpha_\lambda, \alpha_\mu) = 0 \\ (\beta_\lambda, \beta_\mu) = 0. \end{array} \right. \quad (\lambda \gtrless \mu).$$

It is the fundamental idea of this development, that the equations (12) form $n(2n-1)$ simultaneous partial differential equations between $2n$ functions α and β, and the $2n$ independent variables p_i and q_i. Now, in a system of more than $(2n-1)$

partial differential equations between $2n$ functions of $2n$ variables, certain fixed conditions must be satisfied. These conditions are here satisfied and in such a manner that the solution possesses the greatest generality possible for a system of $n(2n-1)$ differential equations. For, we are limited in no way by the integration of the successively appearing partial differential equations, in which the given variables are successively replaced by new ones. This principle, which is also of importance in other branches of mathematics, *e. g.*, in the theory of those differential equations which are fundamental in the theory of invariants,* deserves the closest attention.

Moreover, one solution of the equations (12) is at once obtained by putting $\alpha_\lambda = p_\lambda$, $\beta_\lambda = q_\lambda$, $(\lambda = 1, 2, \ldots n)$.

These results can be at once applied to the integrals of the differential equations (9), § 10, after introducing a new variable τ, making $(2n+2)$ variables in all, and putting

$$H' = H + \tau,$$

so that H' contains only p_i, q_i and t.

Further, for uniformity, introduce the expressions $t = q_0$, $t = p_0$, and consider H' as the function α of p and q which is taken at will in (3). Then (3) becomes

$$1 = (H', A) = \sum_{i=0}^{i=n}\left(\frac{\partial H'}{\partial p_i}\frac{\partial A}{\partial q_i} - \frac{\partial H'}{\partial q_i}\frac{\partial A}{\partial p_i}\right)$$

$$= \sum_{i=1}^{i=n}\left(\frac{\partial H}{\partial p_i}\frac{\partial A}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial A}{\partial p_i}\right) - \frac{\partial H}{\partial t}\cdot\frac{\partial A}{\partial \tau} + \frac{\partial A}{\partial t},$$

in which p and q are independent variables and A, a dependant variable. It is at once satisfied by $A = t = q_0$, and we shall take this quantity for β_1 in equation (3).

The equation (4), which may be written,

$$0 = (H', \alpha),$$

has $(2n+1)$ independent solutions. One of them is H' itself

See Arnohold, *Ueber eine fundamentale Begrundung der Invariantentheorie.* Berlin, 1863.

and the others can be so selected as to be independent of τ. For, if we put $\frac{\partial a}{\partial \tau}=0$, the differential equation $0=(a, H')$ becomes

(13) $$0=(a, H)+\frac{\partial a}{\partial t}.$$

This equation is consistent, since its cofficients do not contain τ. It is indeed equation (19), § 10, the one which defined the $2n$ integrals of equations (9). For the given value of A, $(A=t=q_0)$, we also have

$$0=(A, a)=\sum_{i=0}^{i=n}\left(\frac{\partial A}{\partial p_i}\frac{\partial a}{\partial q_i}-\frac{\partial A}{\partial q_i}\frac{\partial a}{\partial p_i}\right).$$

In order to show clearly the fundamental principle of the method, we will, for a moment, put $H'=\alpha_0$, $A=t=\beta_0$, so that $(\alpha_0, \beta_0)=1$, and in general $(\alpha_0, a)=(\beta_0, a)=0$.

From this follows, finally:

If we take a system of $2(n+1)$ variables p_i, q_i, $(i=0,\dots n)$, and determine $\alpha_0, \alpha_1, \dots \alpha_n, \beta_0, \beta_1, \dots \beta_n$ in such a way that they satisfy the conditions (12); then as a special case, we can put $\alpha_0=H'=H+p_0$, $\beta_0=q_0$, where H is independent of p_0 and q_0, but is an arbitrary function of the quantities p and q. The equation $(\alpha_0, \beta_0)=1$ is then immediately satisfied. The remaining functions $\alpha_1, \dots \alpha_n$, $\beta_1, \dots \beta_n$, can be so selected as to be independent of β_0. In the combinations $(\alpha_\lambda, \beta_\lambda)$ of these $2n$ functions, there will be no derivatives with respect to p_0 and q_0. Finally, if we put $p_0=\tau$, $q_0=t$, the functions $\alpha_1, \dots \alpha_n, \beta_1, \dots \beta_n$ satisfy equations (19), §10, for a. They are therefore integrals of (9), § 10, which satisfy the conditions (12) of this section. By reason of (54), (55), (63) and (64) of § 10, Lagrange's combinations $[\alpha_\lambda, \alpha_\mu]$ give

(14) $$\begin{cases} [\alpha_\lambda, \beta_\lambda]=1, \\ [\alpha_\lambda, \beta_\mu]=0, & (\lambda \gtrless \mu), \\ [\alpha_\lambda, \alpha_\mu]=[\beta_\lambda, \beta_\mu]=0. \end{cases}$$

The $2n$ integrals $\alpha_1, \dots \alpha_n, \beta_1, \dots \beta_n$, which satisfy the conditions (12) will hereafter be called a system of canonical inte-

grals, or also, since they do not change with the time t, a system of canonical constants of integration. Among these there is one of extraordinary simplicity. It is obtained by putting $\alpha_i = p_{i'}$, $\beta_i = q_{i'}$, where $p_{i'}$ and $q_{i'}$ denote the initial values of p_i and q_i for a fixed epoch $t' = t$. The equations (14), § 10, here become

$$(15) \qquad \begin{cases} p_i = p_i(p_{1'}, \dots p_{n'}, q_{1'} \dots q_{n'}, t, t') \\ q_i = q_i(p_{1'}, \dots p_{n'}, q_{1'} \dots q_{n'}, t, t') \end{cases}$$

and their solutions for $p_{i'}$ and $q_{i'}$, (15), § 10

$$(16) \qquad \begin{cases} p_{i'} = p_{i'}(p_1 \dots p_n, q_1 \dots q_n, t', t) \\ q_{i'} = q_{i'}(p_1 \dots p_n, q_1 \dots q_n, t', t). \end{cases}$$

The equations (16) are obtained from (15 by interchanging the primed letters with the ones not primed. If in (15) or (16) $t = t'$, they become identical, and, in this case, give.

$$(17) \qquad p_i = p_{i'}, \quad q_i = q_{i'},$$

In Poisson's expression (a, b), we now substitute for a and b, the two functions which form the second members of (16). Since they are independent of t and since no derivatives with respect to t occur in (a, b), t may receive any value t' before differentiation. But then the functions $p_{i'}, q_{i'}$ pass directly into p_i and q_i, and we have

$$(18) \qquad \begin{cases} (p_{\lambda'}, q_{\lambda'}) = 1, \\ (p_{\lambda'}, q_{\mu'}) = 0, \qquad (\lambda \gtrless \mu), \\ (p_{\lambda'}, p_{\mu'}) = (q_{\lambda'}, q_{\mu'}) = 0, \end{cases}$$

therefore, the conditions for a canonical system.

Finally, resuming the unanswered question, will the repeated use of Poisson's theorem afford all the integrals from two given ones, a and b? One of them a may at once be taken $= \alpha_1$; the other one b is then a function of α_i and β_i and equation (11) give, at once,

$$(19) \qquad \begin{cases} (a, b) = \dfrac{\partial b}{\partial \beta_1}, \text{ consequently,} \\ (a, (a, b)) = \dfrac{\partial^2 b}{\partial \beta_1^2}, \\ (a, (a, (a, b))) = \dfrac{\partial^3 b}{\partial \beta_1^3}, \\ \text{etc.} \end{cases}$$

Therefore, if b, taken as a function of β_1, satisfies no differential equation of the $(2n-2)^{\text{th}}$ or lower order which contains a_1 and b only in its coefficients, we get from (19) $(2n-2)$ new integrals which with a and b make the necessary number. If there is such an equation, the combination with b may, or may not, furnish the remaining integrals. In every case there is a possibility, and it is generally true, that two integrals will determine all the others, although there are exceptions which cannot be determined beforehand.

It is in this sense that we are to understand what Jacoby has said in his *Vorlesungen über Mechanik*, (Werke, Supplementband, p. 269). He says:

"That the importance of this theorem (Poisson's), which has so long beeu kuown, has not been recognized, is due to a remarkable circumstance. The cases in which it was used happened to be the very ones in which the newly formed expressions gave no new integral, but in which the resulting expression was identically equal to zero, or equal to a number different from zero, perhaps 1. These cases are exceptions in the general theory, but are very common in the applications. The integral, which, by combination with a second, is to give all the others, must necessarily be one belonging to the special problem. But the first integrals, derived for a given problem, are usually those which depend upon general principles, (such, for example, as the law of areas), and are not those which especially belong to the special problem. Hence we can not expect to obtain the other integrals from them."

It should, however, be noted, that when this integral belongs especially to the special problem, it may not afford all the integrals, although, in the general case this is necessary.

13. The Canonical Constants for the Elliptic Elements of the Orbit of a Planet.

The principles of the preceding paragraphs will now be used in order to obtain a system of six canonical elements to replace the elements a, e, Ω, i, π, ε, given in § 2. The determinant (22),

§ 11, shows that the combination of a with the other elements, except ε, gives 0, and with ε it gives

$$(a, \varepsilon) = -2\sqrt{\frac{a}{\mu}}.$$

The distance a is determined by the equation

$$\frac{1}{2}\frac{(dx^2+dy^2+dz^2)}{dt^2} - \frac{\mu}{r} = -\frac{\mu}{2a}.$$

If we put

(1) $$\alpha_1 = -\frac{\mu}{2a},$$

we get

$$(\alpha_1, \varepsilon) = (a, \varepsilon)\frac{\partial \alpha_1}{\partial a} = -\sqrt{\frac{\mu}{a^3}},$$

and

$$\left(\alpha_1, -\frac{\varepsilon}{\sqrt{\frac{\mu}{a^3}}}\right) = 1.$$

For β_1 we could at once select $-\dfrac{\varepsilon}{\sqrt{\frac{\mu}{a^3}}} = -\dfrac{\varepsilon}{n}$, but instead of this, we shall select the time t_0 of perihelion passage, which, according to (20), § 2, is determined by the equation

(2) $$t_0 = -\frac{\varepsilon}{n} + \frac{\pi}{n} = \beta_1,$$

$$(\alpha_1, \beta_1) = \left(\alpha_1, -\frac{\varepsilon}{n}\right) + \left(\alpha_1, \frac{\pi}{n}\right) = 1,$$

and

$$(\alpha_1, \Omega) = (\alpha_1, i) = (\alpha_1, \pi) = (\alpha_1, \varepsilon) = 0.$$

Further, by (22), § 11, we also have

$$(\beta_1, \Omega) = \left(-\frac{\varepsilon}{n}, \Omega\right) + \left(\frac{\pi}{n}, \Omega\right) = \frac{1}{n}\Big((-\varepsilon, \Omega) + (\pi, \Omega)\Big) = 0,$$

$$(\beta_1, i) = \frac{1}{n}\Big[(-\varepsilon, i) + (n, i)\Big] = 0,$$

$$(\beta_1, \pi) = \frac{1}{n}\Big[(-\varepsilon, \pi) + (\pi, \pi)\Big] = 0,$$

$$(\beta_1, e) = \frac{1}{n}\Big[(-\varepsilon, e) + (\pi, e)\Big] = -\frac{a(1-e^2)}{e\mu}.$$

Therefore, if b is any function of $\alpha_1, e, \Omega, i, \pi$ with the exception of $\quad_1$, it follos from (1), § 12, that

$$(\beta_1, b) = -\frac{\partial b}{\partial \alpha_1} - \frac{\partial b}{\partial e}\frac{a(1-e^2)}{e\mu} = -\frac{\partial b}{\partial \alpha_1} + \frac{\partial b}{\partial e}\frac{(1-e^2)}{2e\alpha_1}.$$

The partial differential equation $(\beta_1, b) = 0$, with b as the function sought and $\alpha, e, \Omega, i, \pi$ as independent variables, has the four independent integrals.

$$b_1 = \frac{1-e^2}{\alpha_1} = -\frac{2a(1-e^2)}{\mu},\ b_2 = \Omega,\ b_3 = i,\ b_4 = \pi.$$

For α_2, b_1 might at once be taken, but instead we shall select the moment of velocity which $= \sqrt{\mu a(1-e^2)} = \sqrt{\mu(-\frac{1}{2}b_1)}$, and hence

(3) $$\alpha_2 = \sqrt{\mu a(1-e^2)} = \sqrt{\mu p}.$$

By reason of (19), § 11,

$$(\alpha_2, \Omega) = 0,\ (\alpha_2, i) = 0,\ (\alpha_2, \pi) = -1.$$

Therefore we might put $\beta_2 = -\pi$, but we shall take

(4) $$\beta_2 = \Omega - \pi,$$

that is, the angle between the ascending node and perihelion.

Then we have

$$(\alpha_2, \beta_2) = 1,$$
$$(\ _2\ , \Omega\ ) = 0,$$
$$(\beta_2, i\ ) = -\frac{\cos i}{\sin i\sqrt{\mu a(1-e^2)}} = -\frac{\cos i}{\alpha_2 \sin i}.$$

If c is a function of α_2, Ω, i,

$$(\beta_2, c) = -\frac{\partial c}{\partial \alpha_2} - \frac{\cos i}{\alpha_2 \sin i}\frac{\partial c}{\partial i}.$$

The partial differential equation $(\beta_2, c) = 0$ in c, with α_2, Ω, i as independent variables, has the two independent integrals, $c_1 = \Omega$, and $c_2 = \alpha_2 \cos i = \sqrt{\mu a(1-e^2)} \cos i$.

Further

$$(c_1, c_2) = 1,$$

hence

(5) $$\alpha_3 = \Omega,$$

(6) $$\beta_3 = \sqrt{\mu a(1-e^2)} \cos i.$$

We have therefore found a canonical system, whose relations to the original elements are expressed by the equations

$$(7)\qquad \begin{cases} \alpha_1 = -\frac{\mu}{2a}, & \beta_1 = t_0 = -\frac{\varepsilon}{n} + \frac{\pi}{n}, \\ \alpha_2 = \sqrt{\mu a(1-e^2)}, & \beta_2 = \Omega - \pi, \\ \alpha_3 = \Omega, & \beta_3 = \sqrt{\mu a(1-e^2)}\cos i. \end{cases}$$

The introduction of the equations (7), gives

$$(\alpha_1, \beta_1) = (\alpha_2, \beta_2) = (\alpha_3, \beta_3) = 1,$$

while all the remaining Poisson's expressions between α and β vanish.

As has been shown, the system (7) is not the only one which can be formed, but, so far as the relations which the new constants bear to the old ones are concerned, it is the simplest.

14. Properties of the Involution Systems.

In § 12, we have seen, that, if $p_1, \ldots p_n, q_1, \ldots q_n$ denote $2n$ independent variables, $2n$ functions of them $\alpha_1, \ldots \alpha_n, \beta_1, \ldots \beta_n$ can be so determined that

$$(1)\qquad (\alpha_\lambda, \beta_\lambda) = 1,$$

$$(2)\qquad (\alpha_\lambda, \beta_\mu) = 0, \qquad (\lambda \gtrless \mu),$$

$$(3)\qquad (\alpha_\lambda, \alpha_\mu) = (\beta_\lambda, \beta_\mu) = 0.$$

Such a system we shall, after Lie, call an involution system.

The simultaneous differential equations (1), (2), (3) to the number of $n(2n-1)$ are to be regarded as the definition of the involution system, and from them we are now to develop its other properties.

In this case the functional determinant is

$$(4)\qquad D = \begin{vmatrix} \frac{\partial \alpha_1}{\partial p_1}, \ldots \frac{\partial \alpha_1}{\partial p_n}, & \frac{\partial \alpha_1}{\partial q_1}, \ldots \frac{\partial \alpha_1}{\partial q_n} \\ \vdots & \vdots \\ \frac{\partial \alpha_n}{\partial p_1}, \ldots \frac{\partial \alpha_n}{\partial p_n}, & \frac{\partial \alpha_n}{\partial q_1}, \ldots \frac{\partial \alpha_n}{\partial q_n} \\ \frac{\partial \beta_1}{\partial p_1}, \ldots \frac{\partial \beta_1}{\partial p_n}, & \frac{\partial \beta_1}{\partial q_1}, \ldots \frac{\partial \beta_1}{\partial q_n} \\ \vdots & \vdots \\ \frac{\partial \beta_n}{\partial p_1}, \ldots \frac{\partial \beta_n}{\partial p_n}, & \frac{\partial \beta_n}{\partial q_1}, \ldots \frac{\partial \beta_n}{\partial q_n} \end{vmatrix}.$$

If we consider the $(2n)^2$ derivatives as independent elements, without reference to their meaning as derivatives, they satisfy the $n(2n-1)$ equations of condition (1), (2), (3), and these equations of condition furnish a series of noteworthy properties of this determinant which possess an unmistakable similarity to the properties of the determinant of an orthogonal substitution. The determinant of an orthogonal substitution has the property that the combination of any two rows or columns gives 0 or 1, according as they are different or not, and there is a somewhat similar relation between D and the determinant

$$(5) \qquad D' = \begin{vmatrix} \frac{\partial \beta_1}{\partial q_1}, \dots & \frac{\partial \beta_1}{\partial q_n}, & -\frac{\partial \beta_1}{\partial p_1}, \dots & -\frac{\partial \beta_1}{\partial p_n} \\ \cdot & & & \cdot \\ \frac{\partial \beta_n}{\partial q_1}, \dots & \frac{\partial \beta_n}{\partial q_n}, & -\frac{\partial \beta_n}{\partial p_1}, \dots & -\frac{\partial \beta_n}{\partial p_n} \\ -\frac{\partial \alpha_1}{\partial q_1}, \dots & -\frac{\partial \alpha_1}{\partial q_n}, & \frac{\partial \alpha_1}{\partial p_1}, \dots & \frac{\partial \alpha_1}{\partial p_n} \\ & \cdot & & \cdot \\ -\frac{\partial \alpha_n}{\partial q_1}, \dots & -\frac{\partial \alpha_n}{\partial q_n}, & \frac{\partial \alpha_n}{\partial p_1}, \dots & \frac{\partial \alpha_n}{\partial p_n} \end{vmatrix}$$

which is derived from D by exchanging the last n rows and columns with the corresponding first rows and columns, and taking the signs as indicated. The combination of the rows of (5) gives, by (2) and (3), 1 when they correspond, and 0 when they do not correspond.

Hence

$$DD' = D^2 = 1,$$

and consequently

$$(6) \qquad D = \pm 1.$$

The analogy may be carried further. If we solve the $2n$ equations

$$(7) \qquad \begin{aligned} \alpha_i &= \alpha_i(p_1, \dots p_n, q_1, \dots q_n), \\ \beta_i &= \beta_i(p_1, \dots \dots p_n, q_1, \dots q_n), \end{aligned}$$

for the p's and q's, we get

$$(8) \qquad \begin{aligned} p_i &= p_i(\alpha_1, \ldots \alpha_n, \beta_1, \ldots \beta_n), \\ q_i &= q_i(\alpha_1, \ldots \alpha_n, \beta_1, \ldots \beta_n), \end{aligned}$$

and, if the functional determinants of (8) take the two forms

$$(9) \qquad \Delta = \begin{vmatrix} \frac{\partial p_1}{\partial \alpha_1}, \ldots, & \frac{\partial q_1}{\partial \alpha_1}, \ldots \\ \vdots & \vdots \\ \frac{\partial p_1}{\partial \beta_1}, \ldots, & \frac{\partial q_1}{\partial \beta_1}, \ldots \\ \vdots & \vdots \end{vmatrix},$$

$$(10) \qquad \Delta' = \begin{vmatrix} +\frac{\partial q_1}{\partial \beta_1}, \ldots, & -\frac{\partial p_1}{\partial \beta_1}, \ldots, \\ \vdots & \vdots \\ -\frac{\partial q_1}{\partial \alpha_1}, \ldots & +\frac{\partial p_1}{\partial \alpha_1}, \ldots \\ \vdots & \vdots \end{vmatrix},$$

then, according to § 12, Δ and Δ' are conjugate determinants with the same symbols as D and D'. It can now be shown that D and Δ' are identical and that D' and Δ are identical. That is, that

$$(11) \qquad \left\{ \begin{aligned} \frac{\partial p_\lambda}{\partial \alpha_\mu} &= \frac{\partial \beta_\mu}{\partial q_\lambda} \\ \frac{\partial p_\lambda}{\partial \beta_\mu} &= -\frac{\partial \alpha_\mu}{\partial q_\lambda} \\ \frac{\partial q_\lambda}{\partial \alpha_\mu} &= -\frac{\partial \beta_\mu}{\partial p_\lambda} \\ \frac{\partial q_\lambda}{\partial \beta_\mu} &= \frac{\partial \alpha_\mu}{\partial p_\lambda} \end{aligned} \right\} \quad (\lambda, \mu = 1, 2, \ldots n).$$

This follows immediately from equations (59), § 10, by substituting in them the expressions given by (62), (63) and (64), and at the same time putting α and β in place of a and b, and then using the equations (1), (2) and (3).

Determinants like D, whose elements satisfy the equations (1), (2) and (3), are called canonical determinants.*

A conclusion of great importance can be obtained from the equations (11). For, if P_1 and P_2 represent any two of the p's and q's, the expression

$$\frac{\partial P_1}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial P_2}+\frac{\partial P_1}{\partial \alpha_2}\cdot\frac{\partial \alpha_2}{\partial P_2}+\ldots+\frac{\partial P_1}{\partial \beta_1}\frac{\partial \beta_1}{\partial P_2}+\ldots$$

is equal to 0 if P_1 and P_2 are different, and to 1 if they are alike. If in this, we substitute for the derivatives of α and β, their values from (11), we obtain at once

$$(12)\qquad \begin{cases} (p_\lambda, q_\lambda)=1, & \\ (p_\lambda, q_\mu)=0, & (\lambda \gtrless \mu), \\ (p_\lambda, p_\mu)=(q_\lambda, q_\mu)=0, & \end{cases}$$

in which the Poisson's expressions are obtained from the earlier ones by replacing α and β by p and q. Hence, the important result:

The solution of an involution system for the primitive variables, gives again an involution system.

Finally we will deduce the following proposition:

If α_i, β_i is an involution system of the original variables p_i, q_i, and we form a second involution system A_i, B_i of α_i, β_i, then the latter is an involution system of p_i and q_i when the expressions α_i, β_i are replaced by p_i, q_i.

Denoting Poisson's expressions by $(A_\lambda, B_\mu)_{\alpha,\beta}$ and $(A_\lambda, B_\mu)_{p,q}$ in which the subscripts α, β, in one case, and p, q in the other indicate the independent variables, we have at once from equation (2), § 12,

$$(13)\qquad (A_\lambda, B_\mu)_{\alpha,\beta}=(A_\lambda, B_\mu)_{p,q},$$

which proves the proposition. And, further, (13) gives the following general result: Poison's expression

$$\sum\left(\frac{\partial \alpha_1}{\partial p_i}\frac{\partial \alpha_2}{\partial q_i}-\frac{\partial \alpha_1}{\partial q_i}\frac{\partial \alpha_2}{\partial p_i}\right)$$

*Canonical Determinants also appear in other problems. See Clebsch and Gordan's *Abel'sche Functionen*, p. 300.

of any two functions a_1, a_2 of p_i, q_i remains unchanged in form, when p_i, q_i are subjected to a canonical substitution.

What precedes is a preparation for our problem, the complete integration of the partial differential equations (1), (2) and (3), and consequently, the determination of the most general form of the canonical system.

We shall proceed to this problem, by first considering n functions

$$(14)\qquad a_i = a_i(p_1, p_2 \ldots p_n, q_1, q_2, \ldots q_n), \quad (i = 1, 2, \ldots n),$$

and the $\frac{1}{2}n(n-1)$ partial differential equations between them

$$(15)\qquad (a_\lambda, a_\mu) = 0.$$

We may assume that these equations represent the condition that the product of the two determinants

$$(16)\quad D_{a,p} = \begin{vmatrix} \frac{\partial a_1}{\partial p_1}, \ldots \frac{\partial a_1}{\partial p_n} \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \frac{\partial a_n}{\partial p_1}, \ldots \frac{\partial a_n}{\partial p_n} \end{vmatrix} \qquad D_{a,q} = \begin{vmatrix} \frac{\partial a_1}{\partial q_1}, \ldots \frac{\partial a_1}{\partial q_n} \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \frac{\partial a_n}{\partial q_1}, \ldots \frac{\partial a_n}{\partial q_n} \end{vmatrix}.$$

obtained by combining the rows, shall be a symmetrical determinant. If, for brevity, we put

$$\frac{\partial a_\lambda}{\partial p_\mu} = p_{\lambda,\mu}, \qquad \frac{\partial a_\lambda}{\partial q_\mu} = q_{\lambda,\mu},$$

the determinants become,

$$(17)\quad D_{a,p} = \begin{vmatrix} p_{1,1}, \ldots p_{1,n} \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ p_{n,1}, \ldots p_{n,n} \end{vmatrix}, \qquad D_{a,q} = \begin{vmatrix} q_{1,1}, \ldots q_{1,n} \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ q_{n,1}, \ldots q_{n,n} \end{vmatrix},$$

and equation (15) becomes

$$(18)\qquad \sum_i p_{\lambda,i}\, q_{\mu,i} = \sum_i p_{\mu,i}\, q_{\lambda,i}.$$

We will now form the minors of (17) and put

$$P_{\lambda,\,\mu} = \frac{1}{D_{a,\,p}} \frac{\partial D_{a,\,p}}{\partial p_{\mu,\,\lambda}}, \qquad Q_{\lambda,\,\mu} = \frac{1}{D_{a,\,q}} \frac{\partial D_{a,\,p}}{\partial q_{\mu,\,\lambda}},$$

and arrange them as determinants.

$$(19) \quad F_p = \begin{vmatrix} P_{1,\,1}, \ldots P_{1,\,n} \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ P_{n,\,1}, \ldots P_{n,\,n} \end{vmatrix}, \qquad F_q = \begin{vmatrix} Q_{1,\,1}, \ldots Q_{1,\,n} \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ Q_{n,\,1}, \ldots Q_{n,\,n} \end{vmatrix},$$

Multiplying (18) by $P_{r,\,\lambda}$, $Q_{\rho,\,\mu}$, where r and ρ are indices selected at will between 1 and n, and letting λ and μ take all possible values, and summing the resulting equations, gives

$$\sum_{i,\,\lambda,\,\mu} p_{\lambda,\,i}\, q_{\mu,\,i}\, P_{r,\,\lambda}\, P_{\rho,\,\mu} = \sum_{i,\,\lambda,\,\mu} p_{\mu,\,i}\, q_{\lambda,\,i}\, P_{r,\,\lambda}\, P_{\rho,\,\mu}.$$

Each three-fold sum can be combined into a single sum. The first member, being summed for λ, gives

$$\sum_{i,\,\mu} (q_{\mu,\,i}\, P_{\rho,\,\mu} \sum_{\lambda} p_{\lambda,\,i}\, P_{r,\,\lambda}).$$

But $\sum\limits_{\lambda} p_{\lambda,\,i} P_{r,\,\lambda}$ is either 1 or 0, according as i is or is not equal to r. Consequently the first member of the above equation becomes

$$\sum_{\mu} q_{\mu,\,r} P_{\rho,\,\mu}.$$

By summing the second member, in the same manner, for μ, we get it

$$= \sum_{\lambda} q_{\lambda,\,\rho}\, P_{r,\,\lambda}$$

Finally, substituting i for λ and μ, λ for r, and μ for ρ, and we obtain

$$(20) \qquad \sum_{i} q_{i,\,\lambda}\, P_{\mu,\,i} = \sum_{i} q_{i,\,\mu}\, P_{\lambda,\,i},$$

and, hence the following interesting proposition of determinants:

If a determinant ($D_{a,\,q}$) has the property, that the product, formed by combining its rows with those of a second, is a symmetrical determinant, then the product formed by combining its columns with the rows of the conjugate to the second, is also symmetrical determinant.

This proposition has an important application to our differential equations. If we consider q as only a parameter in (14), then $D_{a,p}$ is a functional determinant. If (14) are now solved for the p's so that they are expressed in terms of a and q, we get

$$(21) \qquad p_i = p_i(a_1, a_2, \ldots a_n, q_1, q_2, \ldots q_n) \qquad (i = 1, 2, \ldots n),$$

then F_p is the functional determinant of (21), and

$$(22) \qquad P_{\lambda,\mu} = \left(\frac{\partial p_\lambda}{\partial a_\mu}\right).$$

The derivatives are enclosed in parentheses to distinguish them from (9) and (10), from which they are evidently entirely different. The substitution of (14) in (21) gives identically $p_i = p_i$. If (21) is differentiated in this sense with reference to $q_{i'}$, it follows that

$$0 = \left(\frac{\partial p_i}{\partial q_{i'}}\right) + \sum_\lambda \left(\frac{\partial p_i}{\partial a_\lambda}\right)\frac{\partial a_\lambda}{\partial q_{i'}}.$$

If the indices i and i' are interchanged, we get

$$0 = \left(\frac{\partial p_{i'}}{\partial q_i}\right) + \sum_\lambda \left(\frac{\partial p_{i'}}{\partial a_\lambda}\right)\frac{\partial a_\lambda}{\partial q_i}.$$

According to (20), the sums in these equations are equal. Hence

$$(23) \qquad \left(\frac{\partial p_i}{\partial q_{i'}}\right) = \left(\frac{\partial p_{i'}}{\partial q_i}\right).$$

These equations show that the second members of (21) are partial derivatives with respect to q of a function

$$(24) \qquad V = V(a_1, a_2, \ldots a_n, q_1, q_2, \ldots q_n),$$

and that

$$(25) \qquad p_i = \frac{\partial V}{\partial q_i} \qquad (i = 1, 2, \ldots n).$$

Since equations (15) follow from equations (23), it follows that equations (15) represent the necessary and sufficient conditions that the solutions of (14) for p should give the latter as a

partial derivitive of V with respect to q. But we may also exchange p and q; (14) then becomes

(26) $$q_i = q_i(a_1, a_2, \ldots a_n, p_1, p_2, \ldots p_n),$$

and,

(27) $$q_i = \frac{\partial W}{\partial p_i}, \quad (i = 1, 2, \ldots n).$$

Equations (27) must be the reverse of equations (25), and hence it follows that

If p_i represents the partial derivatives of a function V with respect to q_i, and if q_i be expressed in terms of p_i, then q_i is the partial derivative of a function W with respect to p_i. This function W is easily determined and upon it the so-called Legendre substitution depends.

If the first set of equations (8) are solved for β, we get

(28) $$\beta_i = \beta_i(a_1, a_2, \ldots a_n, p_1, p_2, \ldots p_n),$$

and since $(p_\lambda, p_\mu) = 0$, the function is

(29) $$W' = W'(a_1, a_2, \ldots a_n, p_1, p_2, \ldots p_n),$$

and is such that

(30) $$\beta_i = \frac{\partial W'}{\partial a_i}.$$

We may now go further. If in (26), we form the derivatives $\left(\frac{\partial q_i}{\partial a_{i'}}\right)$, we get the quantities $Q_{i', i}$, which are the minors of the system $D_{a, q}$ in (17). And if, in (28), we form the derivatives $\left(\frac{\partial \beta_{i'}}{\partial p_i}\right)$, they are likewise the minors of the system

$$\begin{vmatrix} \frac{\partial p_1}{\partial \beta_1}, \ldots \frac{\partial p_n}{\partial \beta_1} \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ \cdot \qquad \\ \frac{\partial p_1}{\partial \beta_n}, \ldots \frac{\partial p_n}{\partial \beta_n} \end{vmatrix}.$$

But, by equations (11), this determinant passes at once into $D_{a, q}$ except that each element must be taken with a negative sign. Hence,

$$\left(\frac{\partial q_i}{\partial \alpha_{i'}}\right) = -\left(\frac{\partial \beta_{i'}}{\partial p_i}\right),$$

and by (27) and (30)

$$\frac{\partial^2(W+W')}{\partial p_i \partial \alpha_{i'}} = 0 \qquad (i, i' = 1, 2, \ldots n).$$

Therefore, $W + W'$ is of the form

$$W + W' = \varphi_1(p_1, \ldots p_n) + \varphi_2(\alpha_1, \ldots \alpha_n),$$

or,

$$W - \varphi_2(\alpha_1, \ldots \alpha_n) = -(W' - \varphi_1(p_1, \ldots p_n)).$$

Since in (27), the differentiation is only with respect to p, we may subtract φ_2 from W, and correspondingly for W'. Hence

(31) $$W' = -W,$$

and we reach our result; viz:

If α_i, β_i are an involution system of p_i and q_i,

(32) $$\alpha_i = \alpha_i(p_1, \ldots p_n, q_1, \ldots q_n),$$
$$\beta_i = \beta_i(p_1, \ldots p_n, q_1, \ldots q_n),$$

and if the β's and q's are expressed in terms of the α's and p's, it follows that

(33) $$q_i = \frac{\partial W}{\partial p_i}, \quad \beta_i = -\frac{\partial W}{\partial q_i},$$

where W denotes a function of the α's and p's,

(34) $$W = W(\alpha_1, \ldots \alpha_n, p_1, \ldots p_n),$$

and, conversely, the equations (33) give an involution system, if they are solved for the α's and β's as unknowns.

This completes the formal integration of equations (1), (2) and (3) so far as it is made to depend upon the selection of some function W and a pure elimination.

(15) Canonical Transformations of the Canonical System of Differential Equations.

We return to the system

(1) $$\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = -\frac{\partial H}{\partial p_i}, \quad (i = 1, 2, \ldots n),$$

and apply to it the so-called *contact transformation* of p_i, q_i into new variables P_i, Q_i,

$$\text{(2)} \qquad \begin{cases} P_i = P_i(p_1, \ldots p_n, \; q_1, \ldots q_n), \\ Q_i = Q_i(p_1, \ldots p_n, \; q_1, \ldots q_n), \end{cases}$$

$$\text{(3)} \qquad \begin{cases} p_i = p_i(P_1, \ldots P_n, \; Q_1, \ldots Q_n), \\ q_i = q_i(P_1, \ldots P_n, \; Q_1, \ldots Q_n), \end{cases}$$

in which (3) are again the solutions of (2). The name of this transformation comes from geometrical considerations and it consists in the use of an involution system for which

$$\text{(4)} \qquad (P_\lambda, Q_\lambda) = 1,$$

and all the remaining Poisson's combinations are equal to zero. From (2) it follows that

$$\begin{aligned} \frac{dP_i}{dt} &= \sum_\lambda \left(\frac{\partial P_i}{\partial p_\lambda} \frac{dp_\lambda}{dt} + \frac{\partial P_i}{\partial q_\lambda} \frac{dq_\lambda}{dt} \right) \\ &= \sum \left(\frac{\partial P_i}{\partial p_i} \frac{\partial H}{\partial q_\lambda} - \frac{\partial P_i}{\partial q_\lambda} \frac{\partial H}{\partial p_i} \right) \\ &= (P_i, H). \end{aligned}$$

If in H, which is a given function of p_i and q_i, the substitution (3) be made, it becomes a function of P_i and Q_i. In this sense, equation (1), §12, gives

$$\begin{aligned} (P_i, H) = \frac{\partial H}{\partial P_1}(P_i, P_1) + \ldots &+ \frac{\partial H}{\partial P_n}(P_i, P_n) \\ + \frac{\partial H}{\partial Q_1}(P_i, Q_1) + \ldots &+ \frac{\partial H}{\partial Q_n}(P_i, Q_n), \end{aligned}$$

or, by (4),

$$\text{(5)} \qquad \frac{dP_i}{dt} = \frac{\partial H}{\partial Q_i}, \quad \frac{dQ_i}{dt} = -\frac{\partial H}{\partial p_i}, \quad (i = 1, 2, \ldots n),$$

where the second equation is obtained in the same manner. From this it follows that *a contact transformation is the only one which leaves the canonical form of equations* (1) *unchanged.* This transformation is of great use in our problem, for its use changes the form of the function H, and in a new form, it may happen that the problem will be easier to handle. The transformation of this paragraph is substantially that given by Jacobi in *Sur l'élimination des noeds dans le problème des trois corps.*

If V denote any given function of p_i and P_i

(6) $$V = V(p_1, \ldots p_n, P_1, \ldots P_n),$$

this transformation follows from the equations

(7) $$q_i = \frac{\partial V}{\partial p_i}, \quad Q_i = -\frac{\partial V}{\partial P_i},$$

as soon as the P's and Q's are computed from them.

A special case occurs when equations (6) contain the P's in linear and entire functions; (7) then gives

$$V = -(P_1 Q_1 + P_2 Q_2 + \ldots + P_n Q_n)$$

and, therefore, $Q_1, \ldots Q_n$ are given functions of p_i. And, further it follows from (7), that the q_i's are linear functions of the P_i's having the p_i's as coefficients. From these the P_i's can be obtained and they are linear functions of the q_i's.

As a special case, we shall now assume that the Q's are linear functions of the p's,

(8) $$Q_i = a_{1,i}\, p_1 + a_{2,i}\, p_2 + \ldots + a_{n,i}\, p_n$$

Then from (7) it follows, that

(9) $$q_i = -a_{i,1} P_1 - a_{i,2} P_2 - \ldots - a_{i,n} P_n,$$

and it remains only to solve equations (9) for the P's.

For the problem of three bodies, § 10, we have

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial u_i}, \quad \frac{du_i}{dt} = \frac{\partial H}{\partial x_i}, \ldots$$

where

(10) $$H = -\frac{m_1 m_2}{r_{12}} - \frac{m_2 m_3}{r_{23}} - \frac{m_3 m_1}{r_{31}} + \sum \frac{u_i^2 + v_i^2 + w_i^2}{m_i}.$$

We may, therefore, take the coordinates x_i, y_i, z_i as $p_1, \ldots p_9$, and u_i, v_i, w_i as $q_1, \ldots q_9$. And for $Q_1, \ldots Q_9$ we shall use ξ_α, ξ_β, ξ, η_α, η_β, η, ζ_α, ζ_β, ζ, these being so taken that

(11) $$\begin{cases} \xi_\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3, \\ \xi_\beta = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3, \\ \xi = m_1 x_1 + m_2 x_2 + m_3 x_3, \end{cases}$$

with six other corresponding equations giving η and ζ in terms of y and z.

The constants α and β must satisfy the conditions

$$(12)\qquad \begin{cases} \alpha_1+\alpha_2+\alpha_3=0, \\ \beta_1+\beta_2+\beta_3=0, \end{cases}$$

from which it appears that the expressions

$$(13)\qquad \alpha_1\beta_2-\beta_1\alpha_2, \quad \alpha_2\beta_3-\beta_2\alpha_3, \quad \alpha_3\beta_1-\beta_3\alpha_1$$

have a common value, which we shall take $=1$.

We shall here denote the P_i's corresponding to ξ, η, ζ by λ, μ, ν. The equations (9) then give

$$(14)\qquad \begin{cases} u_1=-\alpha_1\lambda_\alpha-\beta_1\lambda_\beta-m_1\lambda, \\ u_2=-\alpha_2\lambda_\alpha-\beta_2\lambda_\beta-m_2\lambda, \\ u_3=-\alpha_3\lambda_\alpha-\beta_3\lambda_\beta-m_3\lambda. \end{cases}$$

If we put,

$$(15)\qquad M=m_1+m_2+m_3,$$

it follows, by reversing and using (12) and (13), that

$$(16)\qquad Mx_1=\xi+\xi_\alpha(\beta_2m_3-\beta_3m_2)-\xi_\beta(\alpha_2m_3-\alpha_3m_2),$$

with eight other analagous equations. From these

$$(17)\qquad (x_1-x_2)=-\xi_\alpha\beta_3+\xi_\beta\alpha_3, \text{ etc.}$$

Also

$$(18)\qquad \begin{aligned} r_{12}^2 &= (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2 \\ &= (\xi_\alpha^2+\eta_\alpha^2+\zeta_\alpha^2)\beta_3^2-2(\xi_\alpha\xi_\beta+\eta_\alpha\eta_\beta+\zeta_\alpha\zeta_\beta)\beta_3\alpha_3 \\ &\quad +(\xi_\beta^2+\eta_\beta^2+\zeta_\beta^2)\alpha_3^2. \end{aligned}$$

In the same way we may form r_{23}^2 and r_{31}^2 and, hence, the six distances are entirely independent of ξ, η, ζ.

If the six constants satisfy the new equation

$$(19)\qquad \frac{\alpha_1\beta_1}{m_1}+\frac{\alpha_2\beta_2}{m_2}+\frac{\alpha_3\beta_3}{m_3}=0,$$

and if, for brevity, we put

$$(20)\qquad \begin{cases} \dfrac{1}{m_\alpha}=\dfrac{\alpha_1^2}{m_1}+\dfrac{\alpha_2^2}{m_2}+\dfrac{\alpha_3^2}{m_3}, \\[2ex] \dfrac{1}{m_\beta}=\dfrac{\beta_1^2}{m_1}+\dfrac{\beta_2^2}{m_2}+\dfrac{\beta_3^2}{m_3}, \end{cases}$$

the kinetic energy becomes, by (14),

$$(21)\qquad \sum \frac{u_i^2+v_i^2+w_i^2}{m_i} = \frac{\lambda_\alpha^2+\mu_\alpha^2+\nu_\alpha^2}{m_\alpha} + \frac{\lambda_\beta^2+\mu_\beta^2+\nu_\beta^2}{m_\beta} + M(\lambda^2+\mu^2+\nu^2).$$

The function H falls therefore into two parts,

$$(22)\quad H' = -\frac{m_1m_2}{r_{12}^2} - \frac{m_2m_3}{r_{23}^2} - \frac{m_3m_1}{r_{31}^2} + \frac{\lambda_\alpha^2+\mu_\alpha^2+\nu_\alpha^2}{2m_\alpha} + \frac{\lambda_\beta^2+\mu_\beta^2+\nu_\beta^2}{2m_\beta},$$
$$H'' = \tfrac{1}{2}M(\lambda^2+\mu^2+\nu^2).$$

Hence, for $\xi, \eta, \zeta, \lambda, \mu, \nu$, we get the system of the sixth order

$$(23)\qquad \frac{d}{dt} = M\lambda, \quad \frac{d\lambda}{dt} = 0, \text{ etc.},$$

which can at once be integrated. For the twelve other variables, we can limit H to the part H' and obtain a system of the twelfth order,

$$(24)\qquad \frac{d\xi_\alpha}{dt} = \frac{\partial H'}{\partial\lambda_\alpha}, \quad \frac{d\lambda_\alpha}{dt} = -\frac{\partial H'}{\partial\xi_\alpha}, \text{ etc.}$$

The system (23) can be at once integrated and gives propositions concerning the center of gravity. The system (24) furnishes the relative motions of the points, and is a complete canonical system of the twelfth order. It is worthy of notice that there are only four equations between the six constants α and β and that some discretion is also possible in their selection, which under some circumstances proves serviceable.

Four integrals of the system (24) are known, those giving the kinetic energy and the propositions concerning areal velocity. By introducing these in (24), it is possible to reduce these equations to other ones, also having the canonical form, but with only eight variables and t. Finally by eliminating t and another variable in this new system, a canonical system of the sixth order can be obtained, but the symmetry disappears. This system has been formed in many ways, but with difficulty in retaining the canonical form, and the function H on which everything depends, losing its original simplicity.

This process has not given us any further reduction, for Lagrange had reduced the problem to the same order. And Lie, by the more complete representations of the the theory of groups, has shown that further reduction of the known integrals is impossible.

16. The Partial Differential Equation of Hamilton and Jacobi.

In the integration of the systems

$$(1) \qquad \frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = -\frac{\partial H}{dp_i}, \quad (i = 1, 2, \ldots n)$$

$2n$ constants, $\alpha_1, \alpha_2, \ldots \alpha_n, \beta_1, \beta_2, \ldots \beta_n$ are introduced.

In § 13, it was shown that these constants can always be selected such functions of p_i and q_i, which may also contain t,

$$(2) \qquad \begin{cases} \alpha_i = \alpha_i(p_1, \ldots p_n, q_1, \ldots q_n, t), \\ \beta_i = \beta_i(p_1, \ldots p_n, q_1, \ldots q_n, t), \end{cases}$$

as to form an involution system. From § 14, it follows that, if α and p be regarded as the independent variables of (2), and if β and q be expressed in terms of them, the resulting equations take the from

$$(3) \qquad q_i = \frac{\partial W}{\partial p_i}, \quad \beta_i = -\frac{\partial W}{\partial \alpha_i}, \quad (i = 1, 2, \ldots n),$$

where W denotes some function of p, α and t,

$$(4) \qquad W = W(p_1, \ldots p_n, \alpha_1, \ldots \alpha_n, t).$$

We shall now examine the function W more closely. The first system of equations (3) furnish n integrations of the system (1) with n arbitrary constants α. If we substitute the values of q_i in

$$(5) \qquad H = H(p_1, \ldots p_n, q_1, \ldots q_n, t),$$

it becomes

$$(6) \qquad H = H\left(p_1, \ldots p_n, \frac{\partial W}{\partial p_i}, \ldots \frac{\partial W}{\partial p_n}, t\right).$$

If we also substitute the values of q_i in the first of the systems (1), we get a system of n differential equations in which p_i and t are the only unknown quantities. If the same substitution is made in the second of systems (1), a similar set of differential equations, between p_i and t, is obtained, which is not independent of the first system. Now, the total differentiation of (3) with respect to t, by the aid of (4), gives

$$\frac{dq_i}{dt} = \frac{\partial^2 W}{\partial p_i \partial t} + \sum_\lambda \frac{\partial^2 W}{\partial p_i \partial p_\lambda} \frac{dp_\lambda}{dt},$$

or, by (1),

$$-\frac{\partial H}{\partial p_i} = \frac{\partial^2 W}{\partial p_i \partial t} + \sum_\lambda \frac{\partial^2 W}{\partial p_i \partial p_\lambda} \frac{\partial H}{\partial q_\lambda},$$

therefore

$$\frac{\partial^2 W}{\partial p_i \partial t} + \frac{\partial H}{\partial p_i} + \sum \frac{\partial H}{\partial q_\lambda} \left(\frac{\partial q_\lambda}{\partial p_i}\right) = 0,$$

where, again, the values of q_λ are to be substituted in $\left(\frac{\partial q_\lambda}{dp_i}\right)$.

In the formation of $\frac{\partial H}{\partial p_i}$, the expression (5) is fundamental. But we may differentiate (6) with respect to p_i, which here appears doubly inasmuch as it is also contained implicitly in W, and if we denote the partial derivatives of (6) with respect to p_i by $\left(\frac{\partial H}{\partial p_i}\right)$, in order to distinguish them from the preceding, it follows that

$$\left(\frac{\partial H}{\partial p_i}\right) = \frac{\partial H}{\partial p_i} + \sum \frac{\partial H}{\partial q_\lambda} \frac{\partial q_\lambda}{\partial p_i},$$

and by the introduction of the former equation, this takes the simple form

(7) $$\frac{\partial \left(\frac{\partial W}{\partial t} + H\right)}{\partial p_i} = 0, \quad (i = 1, 2, \ldots n),$$

where H has the form given by (6). The equation (7) shows that the expression

(8) $$\frac{\partial W}{\partial t} + H = A$$

is entirely indedendent of p_i. We can further show that it does not contain α_i. Differentiating (8) with respect to α_i, we get

$$\frac{\partial A}{\partial \alpha_i} = \sum_\lambda \frac{\partial H}{\partial q_\lambda} \frac{\partial q_\lambda}{\partial \alpha_i} + \frac{\partial^2 W}{\partial \alpha_i \partial t} = \sum \frac{\partial p_\lambda}{\partial t} \frac{\partial \left(\frac{\partial W}{\partial \alpha_i}\right)}{\partial p_\lambda} + \frac{\partial \left(\frac{\partial W}{\partial \alpha_i}\right)}{\partial t}.$$

But the second member of this is $=0$, since it is the total derivative of $\frac{\partial W}{\partial \alpha_i}$, that is of a constant with respect to the time. Hence

$$(9) \qquad 0=\frac{\partial\left(\frac{\partial W}{\partial t}+H\right)}{\partial \alpha_i}.$$

The equations (7) and (9) show that (8) is a pure function of the time with no arbitrary constants. But, since the derivative of this function $f(t)$ vanishes in the differential equations (1), it may be subtracted from H, and then

$$(10) \qquad 0=\frac{\partial W}{\partial t}+H\left(p_1, \ldots p_n, \frac{\partial W}{\partial p_1}, \ldots \frac{\partial W}{\partial p_n}, t\right).$$

This equation, in which p and t are the independent variables and W the function to be determined, is the *partial differential equation of Hamilton and Jacobi.*

Jacobi has shown that its complete integral immediately furnishes the integrals of the system (1). This will now be proven.

By a complete integral, or a complete solution of (10), is meant, according to Lagrange, a solution which has as many independent arbitrary constants as there are independent variables. The number in this case is $n+1$. One of these is a constant belonging to W and this may be neglected. Let the remaining n constants be $\alpha_1, \alpha_2, \ldots \alpha_n$, so that W has the form

$$(11) \qquad W=W(p_1, \ldots p_n, \alpha_1, \ldots \alpha_n, t).$$

Introducing

$$(12) \qquad q_i=\frac{\partial W}{\partial p_i},$$

we can at once give the integrals of the system

$$(13) \qquad \frac{dp_i}{\partial t}=\frac{\partial H}{\partial q_i},$$

in which the value of q_i, obtained by differentiating (12), is to be used. (In the second member of (13), p and t represent the variables and the α's the parameters). If (10) is differentiated

with respect to a parameter α_i, it follows, since it is contained only in W, that

$$0 = \frac{\partial\left(\frac{\partial W}{\partial \alpha_i}\right)}{\partial t} = \sum_\lambda \frac{\partial H}{\partial q_\lambda} \frac{\partial\left(\frac{\partial W}{\partial \alpha_i}\right)}{\partial p_\lambda},$$

and hence, by (13),

$$0 = \frac{\partial\left(\frac{\partial W}{\partial \alpha_i}\right)}{\partial t} + \sum_\lambda \frac{\partial p_\lambda}{\partial t} \frac{\partial\left(\frac{\partial W}{\partial \alpha_i}\right)}{\partial p_\lambda},$$

or

$$0 = \frac{d\left(\frac{\partial W}{\partial \alpha_i}\right)}{dt}.$$

Hence, the expressions,

$$\beta_i = -\frac{\partial W}{\partial \alpha_i} \tag{14}$$

represent the n integrals of (13), and by solution for p_i will give these as functions of the time t, the arbitrary parameters α, and the constants of integration β.

By the total differentiation of (10) with respect to p_i, we get

$$0 = \frac{\partial\left(\frac{\partial W}{\partial p_i}\right)}{dt} + \sum_\lambda \left(\frac{\partial H}{\partial q_\lambda} \frac{\partial\left(\frac{\partial W}{\partial p_i}\right)}{\partial p_\lambda} \right) + \frac{\partial H}{\partial p_i}$$

$$= \frac{\partial\left(\frac{\partial W}{\partial p_i}\right)}{\partial t} + \sum_\lambda \left(\frac{\partial p_\lambda}{\partial t} \frac{\partial\left(\frac{\partial W}{\partial p_i}\right)}{\partial p_\lambda} \right) + \frac{\partial H}{\partial p_i}$$

$$= \frac{d\left(\frac{\partial W}{\partial p_i}\right)}{dt} + \frac{\partial H}{\partial p_i},$$

or, by (12),

$$\frac{dq_i}{dt} = -\frac{\partial H}{\partial p_i}. \tag{15}$$

Equations (13) and (15) reproduce the complete system (1); and (12) and (14) are its integrals with the $2n$ arbitrary constants α_i and β_i.

The fundamental idea in these investigations consists in the disclosure of the intimate connection between the theory of partial differential equations of the first order, and the theory of simultaneous differential equations. This connection, for linear equations and also for equations containing two independent variables, has been known since the time of Lagrange. By a slight generalization, it easily appears, that the one problem can be transformed into the other. The reader is referred to Jacobi's masterly investigations.

17. H Not Containing the Time.

Up to this time, H has been a function of p_i, q_i and t. If it does not contain t, the partial differential equation (10) of the preceding paragraph becomes

$$(1) \qquad 0=\frac{\partial W}{\partial t}+H\left(p_1, \ldots p_n, \frac{\partial W}{\partial p_1}, \ldots \frac{\partial W}{\partial p_n}\right).$$

Since (1) does not explicitly contain t, a peculiar transformation may be used, in which the transformation formulas depend upon the yet undetermined W, instead of given functions. Substitute, in (1), a new variable for t, such that

$$(2) \qquad \frac{\partial W}{\partial t}=-h,$$

$$(3) \qquad t=t\,(p_1, \ldots p_n, h),$$

and for the function W, a new function V, such that

$$(4) \qquad V=W-(t-\varepsilon)\frac{\partial W}{\partial t}=W+(t-\varepsilon)\,h,$$

where ε is a constant selected arbitrarily.

Remembering that

$$(5) \qquad W=W(p_1, \ldots p_n, t),$$

and making the substitution (3) in (4), we have, by differentiation

$$(6) \qquad \frac{\partial V}{\partial h}=\frac{\partial W}{\partial t}\frac{\partial t}{\partial h}-\frac{\partial W}{\partial t}\frac{\partial t}{\partial h}-(t-\varepsilon)\frac{\partial\left(\frac{\partial W}{\partial t}\right)}{\partial h}=+(t-\varepsilon),$$

$$(7) \qquad \frac{\partial V}{\partial p_i} = \frac{\partial W}{\partial p_i} + \frac{\partial W}{\partial t}\frac{\partial t}{\partial p_i} + h\frac{\partial t}{\partial p_i} = \frac{\partial W}{\partial p_i}.$$

Accordingly, equation (1) becomes

$$(8) \qquad -h + H\left(p_1, \ldots p_n, \frac{\partial V}{\partial p_1}, \ldots \frac{\partial V}{\partial p_n}\right) = 0.$$

In this case, $p_1, \ldots p_n$ and h are the independent variables. The derivative of V with respect to h does not occur; hence, this equation can be integrated as if h were constant. In this sense, the complete solution of (8) has in its expression

$$(9) \qquad V = V(p_1, \ldots p_n, h, \alpha_1, \ldots \alpha_{n-1}),$$

$n-1$ arbitrary constants $\alpha_1, \ldots \alpha_{n-1}$ besides the additive one depending on V. The latter, we shall not consider. With equal propriety $\alpha_1, \ldots \alpha_{n-1}$ may be regarded as constants or as arbitrary functions of h. We shall consider them as constants.

Returning to the primitive function W, we have

$$(10) \qquad W = V - (t - \varepsilon)h = V - h\frac{\partial V}{\partial h}.$$

If the substitutions (9) and (7) are made in this, W takes the form

$$(11) \qquad W = W(p_1, \ldots p_n, \alpha_1, \ldots \alpha_{n-1}, t - \varepsilon),$$

so that the earlier constants $\alpha_1, \ldots \alpha_n$, now have the values

$$(12) \qquad \alpha_1, \alpha_2, \ldots \alpha_{n-1}, \varepsilon,$$

Therefore, the quantities (12) are half of a system of canonical constants of integration. The others are determined by the equations (14), § 16, which here, have the form

$$(13) \qquad \beta_i = -\frac{\partial W}{\partial \alpha_i} \qquad (i = 1, 2, \ldots n-1)$$

and

$$(14) \qquad \beta_n = -\frac{\partial W}{\partial \varepsilon} = +\frac{\partial W}{\partial t} = -h.$$

But the derivatives of V in the form (9) can also be at once substituted in (13). For, according to (10),

$$(15) \qquad \frac{\partial W}{\partial \alpha_i} = \frac{\partial V}{\partial \alpha_i} + \frac{\partial V}{\partial h}\frac{\partial h}{\partial \alpha_i} - (t - \varepsilon)\frac{\partial h}{\partial \alpha_i} = \frac{\partial V}{\partial \alpha_i}.$$

Hence the substitution, which changes W into V, need not be taken into consideration, and we get the following final result:

If there is a system of total differential equations

$$\frac{dp_i}{dt}=\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt}=-\frac{\partial H}{\partial p_i}, \quad (i=1, \ldots n), \tag{16}$$

in which H does not contain the time t, the complete integral V of the differential equation (8) *can be formed, in which $\alpha_1, \alpha_2, \ldots \alpha_{n-1}$ are the arbitrary constants. The solution of* (16) *will be given by the formulas*

$$q_i=\frac{\partial V}{\partial p_i}, \quad (i=1, \ldots n),$$

$$\beta_i=-\frac{\partial V}{\partial \alpha_i}, \quad (i=1, \ldots n-1),$$

$$t-\varepsilon=-\frac{\partial V}{\partial h},$$

and the n pairs of canonical constants of integration are

$$\alpha_1, \beta_1; \ \alpha_2, \beta_2; \ldots \alpha_{n-1}, \beta_{n-1}; \ \varepsilon, -h.$$

For the special system (8), § 10, the partial differential equation (8) is

$$h=-\sum\frac{m_\lambda m_\mu}{r_{\lambda,\mu}}+\tfrac{1}{2}\sum\frac{\left(\frac{\partial V}{\partial x_i}\right)^2+\left(\frac{\partial V}{\partial y_i}\right)^2+\left(\frac{\partial V}{\partial z_i}\right)^2}{m_i}, \tag{17}$$

and the equations (13) represent the final equations between the coordinates, while the velocities are at once determined by the formulas

$$m_i\frac{dx_i}{dt}=\frac{\partial V}{\partial x_i}, \quad m_i\frac{dy_i}{dt}=\frac{\partial V}{\partial y_i}, \quad m_i\frac{dz_i}{dt}=\frac{\partial V}{\partial z_i}. \tag{18}$$

The partial differential equation (17) was first obtained by Hamilton, but he did not see the deeper conclusions to be drawn from it, and the way in which he proceeded was quite another. He defined V as the "action," and we shall see how easily his definition may be drawn from (17). If V be determined from (17) as a function of the coordinates, and its total derivative with respect to t be taken, we get

$$(19)\qquad \frac{dV}{dt}=\sum\left(\frac{\partial V}{\partial x_i}\frac{dx_i}{dt}+\frac{\partial V}{\partial y_i}\frac{dy_i}{dt}+\frac{\partial V}{\partial z_i}\frac{dz_i}{dt}\right)$$

$$=\sum m_i\frac{dx_i^2+dy_i^2+dz_i^2}{dt^2}=2\,T,$$

and

$$V=2\int^t T dt+C,$$

and, if we suppose, that V vanishes for a definite epoch t',

$$(20)\qquad V=2\int_{t'}^{t} T dt.$$

The action V is, accordingly, equal to twice the double integral of the kinetic energy with respect to the time. Moreover, this integral can be at once obtained from (21), § 6. In this case $V'=\sum\frac{m_\lambda m_\mu}{r_{\lambda\mu}}$. If we consider only the relative motion of the center of gravity, $C'=h$, and therefore, since $T=V'+h$,

$$(21)\qquad V=2\int_{t'}^{t}(V'+h)dt=\frac{1}{\Sigma m}\cdot\frac{d\Sigma m_\lambda m_\mu r_{\lambda\mu}^2}{dt}\Bigg|_{t'}^{t}-2h(t-t').$$

Finally we will give Hamilton's derivation of the partial differential equation (17), because it is the best known and depends upon a well known principle—the so-called Hamiltonian Principle. It has, however, been shown by Jacobi that this derivation is made with an unnecessary limitation which tends to cover the true relations.

Hamilton started with the equations

$$(22)\qquad m_i\frac{d^2x_i}{dt^2}=\frac{\partial U}{\partial x_i},\quad m_i\frac{d^2y_i}{dt^2}=\frac{\partial U}{\partial y_i},\quad m_i\frac{d^2z_i}{dt^2}=\frac{\partial U}{\partial z_i},$$

$$(23)\qquad U=\sum\frac{m_\lambda m_\mu}{r_{\lambda\mu}}$$

If we multiply the equations (22) in order by the virtual variations δx_i, δy_i, δz_i, and, for brevity, put

(24) $$\frac{dx_i}{\partial t} = u_i, \quad \frac{dy_i}{dt} = v_i, \quad \frac{dz_i}{dt} = w_i,$$

we get

$$\sum m_i\left(\frac{d^2x_i}{dt^2}\delta x_i + \frac{d^2y_i}{dt^2}\delta y_i + \frac{d^2z_i}{dt^2}\delta z_i\right)$$

$$= \sum\left(\frac{\partial U}{\partial x_i}\delta x_i + \frac{\partial U}{\partial y_i}\delta y_i + \frac{\partial U}{\partial z_i}\delta z_i\right) = \delta U.$$

Now

$$\frac{d^2x}{dt^2}\delta x = \frac{d}{dt}\left(\frac{dx}{dt}\delta x\right) - \frac{dx}{dt}\delta\frac{dx}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\delta x\right) - \tfrac{1}{2}\delta\left(\frac{dx}{dt}\right)^2.$$

The preceding equation, therefore, becomes

$$\frac{d}{dt}\sum m_i\left(\frac{dx_i}{dt}\delta x_i + \frac{dy_i}{dt}\delta y_i + \frac{dz_i}{dt}\delta z_i\right)$$

$$= \delta U + \delta T = \delta(U + T),$$

and hence, on integration with respect to t, between the limits t' and t,

(25) $$\sum m_i\left(\frac{dx_i}{dt}\delta x_i + \frac{dy_i}{dt}\delta y_i + \frac{dz_i}{dt}\delta z_i\right)\Bigg|_{t'}^{t}$$

$$= \int_{t'}^{t}\delta(U + T)\,dt = \delta\int_{t'}^{t}(U + T)\,dt.$$

This equation, in which $\delta x_i, \ldots$ denote any selected infinitesimal variations, was Hamilton's starting point.

The coordinates and velocities are functions of the time and of $6n$ constants. For the initial values of these, corresponding to the time t', we shall take $x_i', y_i', z_i', u_i', v_i', w_i'$ so that the final equations take the form

(26) $$x_i = x_i(x_1', y_1', z_1', \ldots u_1', v_1', w_1', \ldots (t - t')).$$

By the differentation of (26) with respect to t, we at once obtain

(27) $$u_i = u_i(x_1', y_1', z_1', \ldots u_1', v_1', w_1', \ldots (t - t')).$$

The action

$$V = 2\int_{t'}^{t} T dt$$

then takes the form

(28) $$V = V(x_1', \ldots u_1', \ldots (t-t')).$$

We will transform (28), by obtaining u', v', w' from (26) and substituting them in it. This gives

(29) $$V = V[x_1, y_1, z_1, \ldots x_1', y_1', z_1', (t-t')].$$

The equation for the kinetic energy is, in this case,

(30) $$h = -\sum \frac{m_\lambda m_\mu}{r_{\lambda\mu}} + \Sigma \tfrac{1}{2} m_i (u_i^2 + v_i^2 + w_i^2).$$

If we substitute for u_i, v_i, w_i their values given by (27) and then for u_i', v_i', w_i' their values given above, h becomes a function of the old and new coordinates and of the time $(t-t')$. If from this we find $(t-t')$ and substitute it in (29), we get, finally,

(31) $$V = V(x_1, y_1, z_1, \ldots x_1', y_1', z_1', \ldots h).$$

This is the form of the equation, which Hamilton took as the basis of his investigations. We shall now assume, that, in (25), the variations δx, δy, δz are not entirely arbitrary, but determined in the following manner. Consider the motion of a system from the time t' to the time t. It is first determined by the coordinates and component velocities. If we suppose that these are changed infinitesimally, we have another motion which is similarly determined. From one instant to the next, the configurations and velocities will change infinitesimally. The changes or variations will be represented by δx_i, δy_i, δz_i, δu_i, δv_i, δw_i, and the initial variations by $\delta x_i'$, $\delta y_i'$, $\delta z_i'$, δu_i, $\delta v_i'$, $\delta w_i'$. Then from (26) and (27), we have δx_i, etc., in the form

$$\delta x_i = \frac{\partial x_i}{\partial x_1'}\delta x_1' + \ldots + \frac{\partial x_i}{\partial u_1'}\delta \mu_1' + \ldots, \text{ etc.}$$

By the aid of (30), the integral in the second member of (25) becomes

$$= \int_{t'}^{t} (2T - h)\, dt = \int_{t'}^{t} 2T dt - h(t - t') = V - h(t - t').$$

With the limitations now made,

$$\delta \int_{t'}^{t} (U + T)\, dt = \delta V - \delta[h(t - t')]$$

$$= \sum\left(\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i + \frac{\partial V}{\partial z_i}\delta z_i\right)$$
$$+ \sum\left(\frac{\partial V}{\partial x_i'}\delta x_i' + \frac{\partial V}{\partial y_i'}\delta y_i' + \frac{\partial V}{\partial z_i'}\delta z_i'\right)$$
$$+ \frac{\partial V}{\partial h}\delta h - (t - t')\,\delta h.$$

It can now be shown, that,

(32) $$\frac{\partial V}{\partial h} = (t - t').$$

By the substitution of the expression

(33) $$(t - t') = (t - t')\ (x_1, \ldots x_1', \ldots h),$$

the element h in (31) is brought back into the form (29).

Therefore,

$$\frac{\partial V}{\partial h} = \frac{\partial V}{\partial (t - t')}\,\frac{\partial (t - t')}{\partial h},$$

where V in the second member is taken from (29).

Now

$$V = 2\int_{t'}^{t} T\, dt = \int_{t'}^{t} (T + U) dt + h(t - t'),$$

consequently,

$$\frac{\partial V}{\partial (t - t')} = T + U + h + (t - t')\,\frac{\partial h}{\partial (t - t')} - P,$$

where

$$P = \sum\left(\frac{\partial \int_{t'}^{t} (T + U)\, dt}{\partial x_i} u_i + \frac{\partial \int \ldots}{\partial y_i} v_i + \frac{\partial \int \ldots}{\partial z_i} w_i\right).$$

Further, it follows from (25), that, if the upper limit t changes, so that the final variations δx_i, δy_i, δz_i become $u_i dt$, $v_i dt$, $w_i dt$,

$$2T = P,$$

and hence

$$\frac{\partial V}{\partial(t-t')} = T+U+h+(t-t')\frac{\partial h}{\partial(t-t')}-2T$$
$$= (t-t')\frac{\partial h}{\partial(t-t')},$$

and, consequently,

$$\frac{\partial V}{\partial h} = \frac{\partial V}{\partial(t-t')}\,\frac{\partial(t-t')}{\partial h}$$
$$= (t-t')\frac{\partial h}{\partial(t-t')}\,\frac{\partial(t-t')}{\partial h} = t-t'.$$

The last two terms of the equation preceding (32), therefore $=0$, and by substitution in (25), we get

$$(34)\quad \Sigma m_i(u_i\delta x_i + v_i\delta y_i + w_i\delta z_i) - \Sigma m_i(u_i'\,\delta x_i' + v_i'\,\delta y_i' + w_i'\,\delta z'_i)$$
$$= \sum\left(\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i + \frac{\partial V}{\partial z_i}\delta z_i\right)$$
$$+ \sum\left(\frac{\partial V}{\partial x_i'}\delta x'_i + \frac{\partial V}{\partial y'_i}\delta y_i' + \frac{\partial V}{\partial z_i'}\partial z_i'\right).$$

Now, the initial variations of the coordinates and component velocities were taken at will, consequently the number of initial and final coordinates may be taken the same. Therefore, for every value of $\delta x_i, \ldots \delta x_i' \ldots$, the equation (34) must consist of two parts

$$(35)\qquad \begin{cases} m_i u_i = \dfrac{\partial V}{\partial x_i}, \ldots, \\[2ex] m_i u_i' = -\dfrac{\partial V}{\partial x_i'}, \text{ etc.} \end{cases}$$

If V, the action, as assumed in (31), is a function of the initial and final coordinates and of h, the second system of (35) gives at once by differentiation, $3n$ integrals, and the first system gives the components of the velocities, while (32) determines the time employed. With this limitation the action V, cannot be used for the solution of the problem, formation of

(31) requires that the problem be already solved. The second system of equations (35) can, of course, furnish only $(3n-1)$ equations between the current coordinates x_i, y_i, z_i, and the relation between them may be at once obtained, by noticing that the equation (30), which, by means of the first of equations (35), becomes

$$(36) \qquad h = -\sum \frac{m_\lambda m_\mu}{r_{\lambda\mu}} + \tfrac{1}{2} \sum \frac{\left(\frac{\partial V}{\partial x_i}\right)^2 + \left(\frac{\partial V}{\partial y_i}\right)^2 + \left(\frac{\partial V}{\partial z_i}\right)^2}{m_i},$$

must also subsist for the initial values, or

$$(37) \qquad h = -\sum \frac{m_\lambda m_\mu}{r_{\lambda\mu}'} + \tfrac{1}{2} \sum \frac{\left(\frac{\partial V}{\partial x_i'}\right)^2 + \left(\frac{\partial V}{\partial y_i'}\right)^2 + \left(\frac{\partial V}{\partial z_i'}\right)^2}{m_i}.$$

This last equation may be regarded as a second partial differential equation in V.

18. The Partial Differential Equation of Hamilton and Jacobi for the Motion of the Planets Around the Sun.

In this case, the function H has the value

$$(1) \qquad H = -\frac{\mu}{r} + \frac{u^2 + v^2 + w^2}{2},$$

and the differential equations of the motion are

$$(2) \qquad \frac{dx}{dt} = \frac{\partial H}{\partial u}, \quad \frac{dy}{dt} = \frac{\partial H}{\partial v}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w},$$

$$(3) \qquad \frac{du}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dv}{dt} = -\frac{\partial H}{\partial y}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z},$$

and Jacobi's partial differential equation is

$$(4) \qquad h = -\frac{\mu}{r} + \tfrac{1}{2}\left\{\left(\frac{\partial V}{\partial x}\right) + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\}.$$

It is now necessary to find a solution of (4), which contains two arbitrary constants besides h and a constant depending on V. In his *Vorlesunger über analytische Mechanik*, Jacobi has

shown how this can easily be done by the introduction of elliptic coordinates. We shall, however, here use V in Hamilton's form, that is expressed by h, x, y, z and the initial values of the coordinates, x', y', z',

$$V = V(x, y, z, x', y', z', h). \tag{5}$$

Now, it is clear, that the action V, depends only on

1. Semi-axis major of the conic section, therefore on
$$h = -\frac{\mu}{2a},$$
2. The distance of the initial place from the sun,
$$r' = \sqrt{x'^2 + y'^2 + z'^2},$$
3. The distance of the final place from the sun,
$$r = \sqrt{x^2 + y^2 + z^2},$$
4. The distance of the final place to the initial place,
$$\rho = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}.$$

For, it is easy to show that these four magnitudes are sufficient to determine the form of the conic section described by the planet and also the velocity at each point in the orbit, as well as

$$V = 2\int_{t'}^{t} T dt = \int_{t'}^{t} (u^2 + v^2 + w^2)\, dt.$$

We may, therefore, write V in the simple form

$$V = V(r, r', \rho, h). \tag{6}$$

Consequently,

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \rho}\frac{\partial \rho}{\partial x} = \frac{\partial V}{\partial r}\frac{x}{r} + \frac{\partial V}{\partial \rho}\frac{(x - x')}{\rho}, \text{ etc.}$$

If we use the equation

$$\begin{aligned} x(x - x') + y(y - y') + z(z - z') &= r^2 - (xx' + yy' + zz') \\ &= r^2 + \tfrac{1}{2}(\rho^2 - r^2 - r'^2) \\ &= \tfrac{1}{2}(\rho^2 - (r + r')^2 + 2r(r + r')), \end{aligned}$$

then

$$\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2=\left(\frac{\partial V}{\partial r}\right)^2+\left(\frac{\partial V}{\partial \rho}\right)^2$$
$$+2\frac{\partial V}{\partial r}\frac{\partial V}{\partial \rho}\left(\frac{(r+r')}{\rho}-\frac{1}{r}\frac{(r+r')^2-\rho^2}{2\rho}\right).$$

Therefore the equation (4) takes the form

$$(7)\quad h=\frac{1}{r}\left[-\mu+\tfrac{1}{2}\frac{\partial V}{\partial r}\frac{\partial V}{\partial \rho}\frac{\rho^2-(r+r')^2}{\rho}\right]$$
$$+\left[\tfrac{1}{2}\left\{\left(\frac{\partial V}{\partial r}\right)^2+\left(\frac{\partial V}{\partial \rho}\right)^2\right\}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial \rho}\frac{(r+r')}{\rho}\right].$$

The corresponding differential equation for the initial coordinates, therefore, becomes

$$(8)\quad h=\frac{1}{r'}\left[-\mu+\tfrac{1}{2}\frac{\partial V}{\partial r'}\frac{\partial V}{\partial \rho}\frac{\rho^2-(r+r')^2}{\rho}\right]$$
$$+\left[\tfrac{1}{2}\left\{\left(\frac{\partial V}{\partial r'}\right)^2+\left(\frac{\partial V}{\partial \rho}\right)^2\right\}+\frac{\partial V}{\partial r'}\frac{\partial V}{\partial \rho}\frac{(r+r')}{\rho}\right].$$

In these differential equations r, r' and ρ are the independent variables and V the function to be determined. It is convenient first to integrate the equation (7), and then to specialize the solution in such a way that the condition (8) is satisfied. Since derivatives with respect to r and ρ only occur in (7), r' may be regarded as a constant parameter and the integration effected by the method given by Lagrange for this case. The solution may also be obtained in other ways, for example, by the introduction of new variables. If we put

$$(9)\qquad \begin{aligned}\tfrac{1}{2}(r+r'+\rho)&=p,\\ \tfrac{1}{2}(r+r'-\rho)&=q,\end{aligned}$$

and, therefore,

$$(9a)\qquad \begin{aligned}r&=p+q-r',\\ \rho&=p-q,\end{aligned}$$

we get

$$\frac{\partial V}{\partial r}=\tfrac{1}{2}\left(\frac{\partial V}{\partial p}+\frac{\partial V}{\partial q}\right),\qquad \frac{\partial V}{\partial \rho}=\tfrac{1}{2}\left(\frac{\partial V}{\partial p}-\frac{\partial V}{\partial q}\right),$$

and, therefore,

$$-\mu+\tfrac{1}{2}\frac{\partial V}{\partial r}\frac{\partial V}{\partial \rho}\frac{\rho^2-(r+r')^2}{\rho}$$

$$=-\mu-\left[\left(\frac{\partial V}{\partial p}\right)^2-\left(\frac{\partial V}{\partial q}\right)^2\right]\frac{pq}{2(p-q)}$$

$$=-\frac{pq}{2(p-q)}\left\{\left[\left(\frac{\partial V}{\partial p}\right)^2-\frac{2\mu}{p}\right]-\left[\left(\frac{\partial V}{\partial q}\right)^2-\frac{2\mu}{q}\right]\right\},$$

likewise,

$$\tfrac{1}{2}\left\{\left(\frac{\partial V}{\partial r}\right)^2+\left(\frac{\partial V}{\partial \rho}\right)^2\right\}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial \rho}\frac{(r+r')}{\rho}-h$$

$$=\frac{1}{2(p-q)}\left\{p\left[\left(\frac{\partial V}{\partial p}\right)^2-2h\right]-q\left[\left(\frac{\partial V}{\partial q}\right)^2-2h\right]\right\},$$

If, for brevity, we put

$$\left(\frac{\partial V}{\partial p}\right)^2-\frac{2\mu}{p}-2h=P,$$

$$\left(\frac{\partial V}{\partial q}\right)^2-\frac{2\mu}{q}-2h=Q,$$

the equation (7) becomes,

$$(10)\qquad \frac{1}{(p+q)-r'}\frac{p+q}{2(p-q)}(P-Q)+\frac{1}{2(p-q)}(pP-qQ)=0.$$

A special solution can be obtained at once, by putting $P=Q=0$,

$$(11)\qquad \frac{\partial V}{\partial p}=\pm\sqrt{2}\sqrt{\frac{\mu}{p}+h},$$

$$(12)\qquad \frac{\partial V}{\partial q}=\pm\sqrt{2}\sqrt{\frac{\mu}{q}+h}.$$

These two equations can plainly exist together. If we take the first root positive and the second negative, we can put for V

$$(13)\qquad V=\sqrt{2}\int_q^p\sqrt{\frac{\mu}{x}+h}\,dx=\sqrt{2}\int_{\frac{r+r'-\rho}{2}}^{\frac{r+r'+\rho}{2}}\sqrt{\frac{\mu}{x}+h}\,dx.$$

Since this expression for V is symmetrical with respect to r and r', it also satisfies the partial differential equation (8).

Since it vanishes for $x=x'$, $y=y'$, $z=z'$, it appears that (13) is the action of the system from the initial to the final configurations, and it is in the desired form.

The time required by the planet to describe the arc of the orbit between the two points, is found by (32), § 17,

$$(14)\quad (t-t')=\frac{\partial V}{\partial h}=\frac{1}{\sqrt{2}}\int_{\frac{r+r'-\rho}{2}}^{\frac{r+r'+\rho}{2}}\frac{1}{\sqrt{\frac{\mu}{x}+h}}dx=\frac{1}{\sqrt{2\mu}}\int_{\frac{r+r'-\rho}{2}}^{\frac{r+r'+\rho}{2}}\frac{dx}{\sqrt{\frac{1}{x}-\frac{1}{2a}}}.$$

This is Lambert's Theorem. It was obtained in an entirely different manner on page 23. From it, it follows, that the time required by a planet to pass from one place in its orbit to another, depends only on

1. The semi-axis major, a,
2. The sum of the distances from the sun, $r+r'$,
3. The chord connecting the two places, ρ.

Moreover this time is the same as that required by a planet in falling in a straight line towards the sun, from the distance $\frac{1}{2}(r+r'+\rho)$ to the distance $\frac{1}{2}(r+r'-\rho)$.

The expression for the time is especially simple when $h=0$, that is when the orbit is a parabola. Equation (5), § 3, then gives

$$(15)\qquad t-t'=\frac{2}{3\sqrt{2\mu}}\left[\left(\frac{r+r'+\rho}{2}\right)^{\frac{3}{2}}-\left(\frac{r+r'-\rho}{2}\right)^{\frac{3}{2}}\right].$$

In this limited case, the proposition was known to Euler, (*Miscell. Ber. A. T.*, S. 20); in its most general form, it was discovered by Lambert. For a long time, the proposition was regarded as a curiosity. Its true source was shown by the investigations of Hamilton and Jacobi. The investigations also give noteworthy equations for the orbit and for the component velocities. If for brevity we put

$$(16)\qquad \sqrt{\frac{\mu}{r+r'+\rho}-\frac{\mu}{4a}}=A,\ \sqrt{\frac{\mu}{r+r'-\rho}-\frac{\mu}{4a}}=B,$$

the equations (35), § 17, give

$$u=\frac{\partial V}{\partial x}=\frac{\partial\int_{\frac{r+r'-\rho}{2}}^{\frac{r+r'+\rho}{2}}\sqrt{\frac{2\mu}{x}-\frac{\mu}{a}}\,dx}{\partial x},^{*}$$

$$=A\frac{\partial(r+r'+\rho)}{\partial x}-B\frac{\partial(r+r'-\rho)}{\partial x},$$

and, therefore,

$$(17)\quad\begin{cases}u=A\left(\frac{x}{r}+\frac{x-x'}{\rho}\right)-B\left(\frac{x}{r}-\frac{x-x'}{\rho}\right),\\ v=A\left(\frac{y}{r}+\frac{y-y'}{\rho}\right)-B\left(\frac{y}{r}-\frac{y-y'}{\rho}\right),\\ w=A\left(\frac{z}{r}+\frac{z-z'}{\rho}\right)-B\left(\frac{z}{r}-\frac{z-z'}{\rho}\right),\end{cases}$$

and likewise

$$(18)\quad\begin{cases}u'=A\left(\frac{x'}{r'}+\frac{x'-x}{\rho}\right)-B\left(\frac{x'}{r'}-\frac{x'-x}{\rho}\right),\\ v'=A\left(\frac{y'}{r'}+\frac{y'-y}{\rho}\right)-B\left(\frac{y'}{r'}-\frac{y'-y}{\rho}\right),\\ w'=A\left(\frac{z'}{r'}+\frac{z'-z}{\rho}\right)-B\left(\frac{z'}{r'}-\frac{z'-z}{\rho}\right).\end{cases}$$

If the initial coordinates and component velocities x', y', z', u', v', w', and, consequently, h are given, the equations (18) give three final equations between x, y, z of a very noteworthy form. They are equivalent to two independent equations, since the equation

$$\frac{u'^2+v'^2+w'^2}{2}=h+\frac{\mu}{r'}$$

must be identically satisfied if we substitute for u', v', w', their values from (18). Further if we eliminate A and B from (18), we obtain

$$(19)\qquad 0=x(y'w'-z'v')+y(z'u'-x'w')+z(x'v'-y'u'),$$

* The coordinate x must, in this case, not be confounded with the x under the sign of integration.

which is the equation of a plane passing through the sun. It is not so easy to see from (18), that the orbit is a conic section. It can be shown in the following manner: If we find A and B from the first two of equations (18), and substitute the values in the expression $A^2 - B^2$, we get

$$A^2 - B^2 = r'\rho \frac{[v'(x'-x) - u'(y'-y)](-v'x' + u'y')}{(xy' - yx')^2},$$

and in the same manner two other analogous expressions for $A^2 - B^2$ can be formed. If, for brevity, the successive numerators be represented by λ_1, λ_2, λ_3, and their sum by λ, it appears that

$$A^2 - B^2 = \frac{r'\rho\lambda}{(xy'-yx')^2 + (yz'-zy')^2 + (zx'-xz')^2}.$$

The denominator can be easily transformed into

$$(x^2+y^2+z^2)(x'^2+y'^2+z'^2) - (xx'+yy'+zz')^2$$
$$= \tfrac{1}{4}[(r+r')^2 - \rho^2][\rho^2 - (r-r')^2],$$

We also have, from (16),

$$A^2 - B^2 = -\frac{2\mu\rho}{(r+r')^2 - \rho^2}.$$

Substituting this value in the above equation, and multiplying by $\dfrac{(r+r')^2-\rho^2}{2\rho}$, gives

$$-\mu = \frac{2r'\lambda}{\rho^2 - (r-r')^2},$$

or

$$\rho^2 - r^2 - r'^2 + 2rr^2 + \frac{2r'\lambda}{\mu} = 0.$$

Now, finally

$$\rho^2 - r^2 - r'^2 = -2(xx' + yy' + zz'),$$

and since λ is linear with reference to x, y, z, we obtain the final equation in the form

$$r = ax + by + cz + d$$

which is nothing else than the equation (25), § 1.

It is also not difficult to verify the other equations of the same section, although no one would have succeeded *a priori* in

getting the notable equations (17) and (18) from those §1. This example shows how a general theory may throw a new and surprising light on an old and special problem.

Jacobi has shown that the partial differential equations (4), can be integrated in other ways, thus giving the solution of the problem a great many forms, which could hardly have been discovered in any other way.

We will finally consider a simple geometrical meaning of the action in the case of an elliptic orbit, as it was given in a note in the *Quarterly Journal of Mathematics, 1866.* (*Note on the Action in an Elliptic Orbit*).

If we denote the velocity of the planet P, by V, (see figure on p. 13), then

$$dV = v^2 dt = \frac{ds}{dt} ds.$$

The increment, dS, of the area described by the radius vector to the sun, is

$$dS = \tfrac{1}{2} dt \sqrt{\mu a (1-e^2)},$$

or, otherwise,

$$2dS = fds,$$

where f denotes the perpendicular from the focus F upon the tangent, and therefore,

$$\frac{ds}{dt} = \sqrt{\frac{\mu a (1-e^2)}{f}}.$$

If f' is the perpendicular from the other focus upon the tangent, by a known property of the ellipse,

$$ff' = b^2 = a^2(1-e^2).$$

Hence

$$dV = \frac{\sqrt{\mu a(1-e^2)}}{f} ds = \frac{n}{\sqrt{1-e^2}} f' ds$$
$$= \frac{2n}{\sqrt{1-e^2}} dS',$$

where S' denotes the area described by the radius vector from the other focus. Now, if we assume that the action and this area are reckoned from perihelion, we get

$$V = \frac{2n}{\sqrt{1-e^2}} S'. \tag{20}$$

Hence it appears, that while the area described by the radius vector from the sun is proportional to the time the area described by the radius vector from the other focus is proportional to the action.

19. HISTORICAL SURVEY FOR THE SECOND DIVISION.

After the general integrals in the problem of n bodies had been obtained, and the differential equations of the motion had been brought into their usual forms, no important advance was made for a long time, at least so far as relates to an insight into the nature of the problem. In 1809, Poisson established a theorem, (*Mémoire sur la variation des constants arbitraires dans les questions méchanique*, Journal de l'école polytechnique, Tome VIII, page 266), which contained the germ of great analytical investigations. It enabled him to give forms of great elegance to the formulas for precession and nutation, as well as to those of perturbed motion. He used his formula only as a means of transformation. This afforded him no opportunity to bring it into relation with a broader theory, and it may well be on this account that he did not develop its true analytical character nor use it in adding new integrals to those already known.

It was about the same time that Lagrange published and soon generalized his formula. (*Mémorire sur la théorie des variations des éléments des planètes.* Oeuvres T. VI, page 713). Lagrange applied his formula to the problem of perturbations, and it enabled him to give the equations arising in this theory very elegant forms. He seems to have prized it only on account of its applications to this problem, which was then engaging his attention.

In 1844, Hamilton published a memoir in the Philosophical Transactions, entitled *On a general Method in Dynamics, by which the study of the motion of all free systems of attracting or repelling points is reduced to the search for, and differentiation of, one central relation or characteristic function.* It enriched ana-

lytical mechanics with a new principle,—that of *varying action.* He introduced the idea of the action as the integral of the kinetic energy with respect to the time from one configuration to another, proposing it as a function of the coordinates of these two configurations and a constant depending on the kinetic energy, and he developed two partial differential equations which satisfy it. He also showed that a knowledge of it led to the immediate derivation of the final integrals by a simple differentiation, and he obtained them for $n = 2$. But he did not recognize the fundamental meaning of the differential equations, nor did he know that the complete solution of one alone lead to that of the whole problem. Jacobi, leaving the original definition of the action in the background, was the first to bring the partial differential equation to the front. He then reduced the problem to the integration of a single partial differential equation. He also generalized Hamilton's investigation by freeing the function H, (which in the problem of n bodies had been the difference between the kinetic energy and the potential energy), from all limitations. By this, and especially by the assumption that H may contain the time, he succeeded in giving the formulas an extraordinary symmetry. The partial differential equations were, however, given a somewhat different form and could not be reduced to Hamilton's original form.

Jacobi did not satisfy himself with this. He connected the partial differential equation with Poisson's theorem and, by searching analytical treatment, succeeded in showing that the system of constants, thus obtained is a canonical one, and, especially, he pointed out that the initial elements form a special canonical system. His endeavor was always to show the connection between the most varied analytical investigations. He extended his studies further, and succeeded in creating a great fundamental theory of partial differential equations of the first order, the presentation of which does not fall within our province.

It is easy to apply the general principles to elliptic motion, and it is worthy to note that Lagrange found, for this case, a system of constants which can at once be transformed into a

canonical system. The method which has been given, for changing a given system of constants into a canonical system, is due to Bour.

Many mathematicians, by their labors, have perfected and generalized Jacobi's ideas concerning involution systems and other celebrated problems. Pfaff's problem is one of these. It relates to the reduction of a differential expression $\Sigma f_i dx_i$, in which the f_i are given functions of x, to its simplest form and of its application to certain theories of space. The works of A. Mayer and S. Lie are also to be mentioned. Their methods have here been used only so far as they seemed available, and the author hopes their thorough application of determinant relations is suited to its place.

Dr. Bruns has recently published (in the *Berichten der sächsischen Akademie*, and in *Acta Mathematica*, Bd. 11.) a very important work, in which he rigorously proves that the known integrals exhaust the algebraic ones. It is to the new theory of functions, with its precise definitions and to its clearly defined progress towards general principles, that we must look for an analysis of the general problem, affording a solution as clear as that which Kepler and Newton have given to the special case of two bodies.

THIRD DIVISION.

Theory of Perturbations.

20. The Consideration of the Solar System as a System of n Bodies.

The heavenly bodies are exceedingly numerous, and on this account, the problem of determining the motion of any one would be incapable of solution by reason of its length alone, were it necessary to consider the forces due to the attraction of all of them. These bodies are widely separated, the forces in obedience to which they move, vary directly as their masses and inversely as the squares their distances apart. Hence, bodies which are relatively at very great distances from a given body, influence its motions only to a very slight degree. Consequently, a very close approximation to the exact motions of any body may be obtained by neglecting the effect of the attractions of relatively very remote bodies. The character of this approximation will now be considered.

In accordance with the notation of § 6, let x_1, y_1, z_1, m_1, ... x_n, y_n, z_n, m_n denote the coordinates and masses of n bodies forming a system. Let ξ, η, ζ denote the coordinates of the center of gravity of the system. Besides the n bodies forming the system, let there be an exterior body whose coordinates and mass are X, Y, Z, M, and whose distance L from the center of gravity of the system is much greater than the greatest dimension l of the system. And, further, let the coordinates, when referred to the center of gravity of the system, be denoted by primed letters. Then

$$L^2 = \sqrt{X'^2 + Y'^2 + Z'^2}.$$

The equations of motion of the exterior body are

$$* \quad \frac{d^2X}{dt^2} = -\sum \frac{m_\lambda(X'-x_\lambda')}{(\sqrt{(X'-x_\lambda')^2+(Y'-y_\lambda')^2+(Z'-z_\lambda')^2})^3}.$$

The quantity under the radical sign is equal to

$$L^2-2(X'x'+Y'y'+Z'z')+(x'^2+y'^2+z^2),$$

in which the orders of the second and third terms with reference to the first term are $\frac{l}{L}$ and $\left(\frac{l}{L}\right)^2$. Remembering that

$$\Sigma mx' = \Sigma my' = \Sigma mz' = 0,$$

the use of the binomial theorem gives

$$\frac{d^2X}{dt^2} = -\Sigma m \frac{X'}{L^3} + \delta,$$

in which the order of δ with reference to the first term is $\left(\frac{l}{L}\right)^2$.

Now

$$\sum m \frac{d^2\xi}{dt^2} + M \frac{d^2X}{dt^2} = 0.$$

or

$$\Sigma m \frac{d^2\xi}{dt^2} = -M \frac{d^2X}{dt^2},$$

and the above equation becomes

$$\frac{d^2\xi}{dt^2}\Sigma m = M\Sigma m \frac{(X-\xi)}{(\sqrt{(X-\xi)^2+(Y-\eta)^2+(Z-z)^2})^3} - M\delta.$$

By neglecting terms of the $\left(\frac{l}{L}\right)^2$th and higher orders with respect to the leading term, the following result is obtained:

The motion of the center of gravity of a system of n bodies, due to the attraction of a very distant exterior body, is the same as if the masses of the n bodies were united at their center of gravity.

We shall next investigate the relative motions of the n bodies with respect to their center of gravity. We have

$$m_1 \frac{d^2x_1'}{dt^2} = m_1 \frac{d^2x_1}{dt^2} - m_1 \frac{d^2\xi}{dt^2}$$

$$= \frac{\partial V}{\partial x_1'} + Mm_1 \left(\frac{(X'-x_1')}{(\sqrt{(X'-x_1')^2+(Y'-y_1')^2+(Z'-z_1')^2})^3} - \frac{X'}{L^3} \right)$$

$$+ \frac{Mm\delta}{\Sigma m}.$$

In the first term V again denotes the potential of the system of n bodies. The order of the second term with the reference to the first is

$$\frac{M}{\Sigma m}\left(\frac{l}{L}\right)^3,$$

the third term is of a still higher order. Hence, if M is not very great compared with Σm, we have, very approximately,

A very distant body has no influence upon the relative motions of a system of n bodies about their center of gravity.

Instead of a single exterior body, consider a system of bodies or many systems, whose dimensions are very small compared with their mutual distances. From what precedes, it is easy to see that any such system will attract as if the bodies composing it were united at their center of gravity, and that the relative motions of the bodies composing it will not be disturbed by the other systems.

The solar system, including the sun, the planets, and their satellites, the comets and the smaller bodies revolving around the sun, is such a system. Some of the comets move away from the sun to indefinite distances, but aside from this, the greatest dimension of the solar system is probably less than one four-thousandth of the distance to the nearest fixed star. At such a distance, the influence of a star on the relative motions of the bodies forming the solar system is of the order

$$\frac{m}{\Sigma m}\cdot\frac{1}{64{,}000{,}000{,}000},$$

which is probably not appreciable, It is possible, and indeed probable, that in the course of centuries, the fixed stars may change the direction and velocity of the otherwise uniform motion of the center of gravity of the solar system. This motion is now directed towards the constellation Hercules.

While we may safely neglect the fixed stars as perturbers of the relative motions in the solar system, we can not neglect the fact that the system itself is composed of bodies instead of points. The dimensions of these bodies are, however, very small compared with their mutual distances, and excepting the motions

about their own centers of gravity, they may be regarded as material points. Moreover, the form of the heavenly bodies enhance the approximation. They are nearly spherical or are composed of nearly homogeneous spherical shells. Now, it is rigorously true that a homogeneous sphere or a body composed of homogeneous spherical shells, attracts an exterior material point as if its mass were concentrated at its center of gravity. The departure of the planets from the spherical form is so slight that it sensibly influences the motions of the satellites only. It appears, therefore, that with close approximation, the sun, the planets and their satellites may be treated as material points which have no further influence upon each other than their mutual altraction. At the same time, there are in the system certain secondary systems, formed of the satellites revolving around a planet as a primary. In these cases it will be necessary to test the degree of approximation. We will take at first the sun, earth and moon. The earth is about four hundred times as far from the sun as from the moon. The influence of the sun on the relative motions of the earth and moon, is, therefore,

$$\frac{M}{m}\left(\frac{1}{400}\right)^3,$$

in which M represents the mass of the sun, and m that of the earth and moon together. This is approximately

$$\frac{1{,}000{,}000}{3 \,.\, 64{,}000{,}000} = \frac{1}{192},$$

so that the sun's disturbing effect is small, although its mass is more than 300,000 times the combined masses of the earth and moon. Yet it is entirely too large to be neglected. Indeed, the numerous inequalities in the moon's motions are nearly all due to the sun.

The influence of the sun on the other secondary systems is much less, partly because their dimensions are relatively less, and partly because the primaries have greater masses and are at greater distances from the sun.

The moving force of the sun on the common center of gravity of the earth and moon is

$$\frac{Mm}{R^2}+\delta,$$

R being the distance of the center of gravity from the sun. The order of δ with reference to the first term is

$$\frac{1}{(400)^2}=\frac{1}{160{,}000}.$$

In this case the order is really less than this because the mass of the earth is more than eighty times that of the moon. In order to determine the resulting decrease of the error δ, we will again assume n bodies, with the same notation as before. Then

$$\frac{d^2\xi}{dt^2}=M\Sigma m\frac{X'}{L^3},+\delta',$$

in which δ' can be shown to have the value

$$\delta'=\frac{M}{L^5}\sum\Big(3m(X'x'+Y'y'+Z'z')x'-\tfrac{3}{2}mX'(x'^2+y'^2+z'^2)$$

$$+\frac{15}{2}m\,.\,\frac{(X'x'+Y'y'+Z'z')^2\,.\,X'}{L^2}\Big)+\varepsilon,$$

in which ε contains the terms of order higher than those developed. Now, let one of the bodies, for example the one whose mass and coordinates are m_1, x_1, y_1, z_1, have a mass far exceeding the sum of the masses Σm of the others. Then the part of δ' which depends upon m_1 will be very small compared with the others. For its coordinates will be of the order $\frac{\Sigma m}{m_1}$, and their squares and products of the order $\left(\frac{\Sigma m}{m_1}\right)^2$. If terms of this order be neglected, δ' will be of the order

$$\frac{M\Sigma m}{L^2}\left(\frac{l}{L}\right)^2.$$

In comparison with the principal term, viz., $\frac{M(m_1+\Sigma m)}{L^2}$, this term is of the order

$$\left(\frac{l}{L}\right)\cdot\frac{\Sigma m}{m_1+\Sigma m},$$

or of the order

$$\left(\frac{l}{L}\right)^2\cdot\frac{\Sigma m}{m_1}.$$

Since the mass of the earth is somewhat more than eighty times that of the moon, the approximation on neglecting δ rises from the order $\left(\frac{1}{400}\right)^2$ to the order

$$\left(\frac{1}{400}\right)^2 \cdot \frac{1}{80} = \frac{1}{12{,}800{,}000}.$$

In what follows, by the position of a planet, will be understood the position of the center of gravity of the secondary system of which the planet is the principal member, and by its mass will be meant the combined masses of the bodies forming the system.

The masses of the asteriods and comets, as well as the other bodies circling about the sun, are so small compared with the masses of the sun and major planets, that they may be neglected. These smaller bodies do not sensibly influence the motions of the larger ones, while the latter may, and in many cases do, influence the motions of the former very much.

From the entire solar system, we will select nine bodies, the Sun, Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, and Neptune, and as in the problem of nine bodies, treat them as mathematical points.

21. The Orbits of the Planets. Theory of Absolute Perturbations.

The general integrals and principles of the preceding sections are, notwithstanding their great significance, not sufficient to solve the problem of n bodies in its most general form. On the contrary, on the supposition that the coordinates and velocities are given for a certain instant, the developments are made in ascending powers of t, as proposed by Euler. Let x be any coordinate and x_0 its value for $t = t_0$, then the development takes the form

$$(1) \qquad x = x_0 + (t - t_0)\left(\frac{dx}{dt}\right)_{t=t_0} + \frac{(t-t_0)^2}{1.2}\left(\frac{d^2x}{dt^2}\right)_{t=t_0} + \cdots$$

The coefficients are found by differentiating the differential equations, substituting lower derivatives for higher, and eliminating the latter down to the first order.

Series (1) converges so long as $(t-t_0)$ does not exceed a certain limiting value. This follows from Cauchy's investigation on the convergence of those series which represent functions defined by differential equations. The theory is beautifully given in Briot and Bouquet's *Théorie des fonctions doublement périodiques.*

The series represented by (1) can be used as long as convergent, and a new series must be begun before it ceases to be convergent. A long interval of time must be divided into suitable parts and then, advancing step by step, the coordinates computed for each epoch.

It is evident that this method suffers from many imperfections. The development of the higher derivatives is very laborious; Gasparis has carried them out to $n=4$. Besides it is not certain but that the errors made in every computation of the series may finally, in the aggregate, amount to such a quantity as to make the results, after long intervals of time, entirely illusory. Finally this progressive development would mask any simpler laws which might exist in the nature of the motion. Hence, this method is to be resorted to only when other methods are not available.

Fortunately, in the case of the solar system, two other methods are known, developed by Euler and Clairaut and so perfected by Lagrange and Laplace as to leave nothing for later mathematicians to do so far as the ground-work of the theory is concerned. The unexpectedly rich results afforded by the use of these methods are due to certain important circumstances which gave rise to their development. The first of these is found in the controlling position of the sun. It is due to its predominating mass, which is upwards of seven hundred times as great as that of all the planets. Astronomers, therefore, except Lagrange in part, have referred the relative motions not to the center of gravity of the system, but to that of the sun. If we follow them, we must employ equations (28), §6.

The differential equations to be used here, are of the form

$$(2) \qquad * \qquad \frac{d^2x_\lambda}{dt^2}=-\frac{\mu_\lambda x_\lambda}{r_\lambda^{\,3}}+\frac{\partial R_\lambda}{\partial x_\lambda}, \quad (\lambda=1,2,\ldots n),$$

in which

(3) $$R_\lambda = \sum_a \frac{m_a}{\sqrt{(x_\lambda - x_a)^2 + (y_\lambda - y_a)^2 + (z_\lambda - z_a)^2}}$$

$$- \sum_a m_a \frac{x_\lambda x_a + y_\lambda y_a + z_\lambda z_a}{\left(\sqrt{(x_a)^2 + (y_a)^2 + (z_a)^2}\right)^3}$$

$$= \sum_a \frac{m_a}{r_{\lambda a}} - \sum m_a \frac{x_\lambda x_a + y_\lambda y_a + z_\lambda z_a}{r_a^3},$$

in which a may have any value, not λ, from 1 to n.

(4) $$\mu_\lambda = M + m_\lambda.$$

In the differential equations (2), the first terms in the second member contains the sun's mass, and is very much larger than the second terms containing R. The first approximation may, therefore, be obtained by neglecting $R_1, R_2, \ldots R_n$. The equations (2) then become the same as those treated in §1, etc., and as the quantities μ differ from each other, and in consequence Kepler's three laws are very approximately true. In fact these laws so perfectly represent the motions of the planets, that Kepler was able to deduce them from observations. Yet the influence of the quantities $R_1, R_2, \ldots R_n$ is appreciable in the course of time, and for this reason Kepler's laws are not rigorously true. These quantities are called the perturbing functions and the following investigations are devoted to them.

If in R, we write $am_2, am_2, \ldots am_n$ for $m_1, m_2, \ldots m_n$ respectively where a is any selected number (Cauchy's Régulateur), then R will be multiplied by a and equations (2) become

(5) * $$\frac{d^2x_1}{dt^2} = -\frac{u_1 x_1}{r_1^3} + a \cdot \frac{\partial R_1}{\partial x_1}, \text{ etc.}$$

If in these equations a be taken as an analytical magnitude the coordinates which satisfy these differential equations, will contain, besides a, $6n$ constants of integration. If $a = 0$, these equations reduce to the forms given in §1, etc.

The essential principle upon which the first method—that of absolute perturbations—developed by mathematicians depends, consists in the assumption that the coordinates in (5), *which are*

functions of α and of the $6n$ constants of integration, can be developed in ascending powers of α.

Any coordinate x may, therefore, be represented in the form

$$x = x^0 + \alpha \delta x^0 + \frac{\alpha^2}{2!}, \delta^2 x^0 + \ldots \tag{6}$$

If $\alpha = 0$, this gives, naturally, the coordinates of Kepler's ellipse. It remains, therefore, to fix the remaining terms $\delta x^0, \delta^2 x^0, \ldots$ of the series (6). If, finally, we put $\alpha = 1$, we obtain the actual coordinates in terms of the time in the form

$$x = x^0 + \frac{\delta x^0}{1} + \frac{\delta^2 x^0}{2!} + \frac{\delta^3 x^0}{3!} + \ldots \tag{7}$$

The determination of $\delta x^0, \delta^2 x^0, \ldots$, which are called the absolute perturbations of the first, second, etc., degrees, can be found by a process of induction, for *the perterbations of the first degree are deduced from the original expressions x^0, etc.; then those of the second degree from the first and the original expressions*; *and, in general, those of any degree from all the lower degrees with the original expressions.*

If we substitute the coordinates given by (6) in (5), the latter become completely identical. Therefore, if we put

$$* \qquad x_1 = x_1{}^0 + \alpha \delta x_1{}^0 + \frac{\alpha^2}{2!} \cdot \delta^2 x_1{}^0 + \ldots \tag{8}$$

for $\alpha = 0$, we again obtain, as must be the case,

$$* \qquad \frac{d^2 x_1{}^0}{dt^2} = -\frac{\mu_1 x_1{}^0}{(r_1{}^0)^3}. \tag{9}$$

$x_1{}^0, y_1{}^0, z_1{}^0$ are determined as in §1.

If now, we take α so small that the developments of both members of (5) in ascending powers of α converge, the coefficients of the equal higher powers in the two members, must be equal. In the first of equations (5), the coefficient of $\frac{\alpha^p}{p!}$ in the first member

$$= \frac{d^2 \delta^p x_1}{dt^2}. \tag{10}$$

The term $-\frac{\mu_1 x_1}{r_1{}^3} = -\frac{\mu_1 x_1}{(\sqrt{x_1{}^2 + y_1{}^2 + z_1{}^2})^3}$ gives as the coeffi-

cient of $\frac{a^p}{p!}$ an expression whose form is somewhat complicated, but whose formation is easy by the help of the polynomial theorem. But it can be seen at once that that part of the coefficient which depends on the perturbations of the p^{th} order, is at once given by

$$(11) \qquad -\mu_1 \frac{\delta^p x_1}{(r_1^0)^3} + \frac{3\mu_1 x_1^0}{(r_1^0)^5}(x_1^0\delta^p x_1 + y_1^0\delta^p y_1 + z_2^0\delta^p z).$$

As to the term $a\frac{\partial R_1}{\partial x_1}$, we can, because of the factor a reach the coefficient of $\frac{a^p}{p!}$ by taking the series (8) up to the $(p-1)^{\text{th}}$ degree. If we denote the sum of these coefficients and that part of the preceding which depends only upon the perturbations to the $(p-1)^{\text{th}}$ degree, by X_1^p, Y_1^p, Z_1^p, and if we write simply x_1, y_1, z_1 for x_1^0, y_1^0, z_1^0, we get

$$(12) * \quad \frac{d^2\delta^p x_1}{dt^2} = -\mu_1 \frac{\delta^p x_1}{r_1^3} + 3\mu \frac{x_1}{r_1^5}(x_1\delta^p x_1 + y_1\delta^p y_1 + z_1\delta^p z_1) + X_1^p.$$

If the original coordinates and the perturbations are known up to and including the $(p-1)^{\text{th}}$ degree, then X_1^p, Y_1^p, Z_1^p are known functions of the time and it is a question of the integration of the differential equations (12) in which $\delta^p x_1$, $\delta^p y_1$, $\delta^p z_1$ are the unknown quantities.

It is to be noted that the successive differential equations to be integrated have always the same general form, and when the general form of the solution has been obtained on the single supposition that X_1^p, Y_1^p, Z_1^p are given functions of the time, we get, by the uniform repetition of the same process all the perturbations and also the coordinates as functions of the time. Of course the resulting series must always be converging.

Since a appeared in the equations (5) as a factor of the masses $m_1, m_2, \ldots$, the quantities $\delta^p x, \ldots$ become homogeneous functions of the p^{th} degree, (at least, we can so represent them), and *the degree of the perturbations is that in which the disturbing masses enter their expressions.*

22. Solution of the Differential Equations for the Absolute Perturbations.

The question now relates to the integration of linear differential equations of the general form

$$(1) \quad * \quad \frac{d^2\delta x}{dt^2} = -\mu\frac{\delta x}{r^3} + \frac{3\mu x}{r^5}(x\delta x + y\delta y + z\delta z) + X,$$

in which x, y, z are such given functions of t as satisfy the differential equations

$$(2) \quad * \quad \frac{d^2x}{dt^2} = -\frac{\mu x}{r^3}.$$

They, therefore, contain the six constants $a, e, \Omega, i, \pi, \varepsilon$, (16), § 2. By hypothesis, X, Y, Z are likewise known functions of t. According to the theory of linear differential equations, the equations (1) are integrated on the assumption that X, Y, Z are zero. When this is done, there are at once six known particular integrals of the system, viz.: the six derivatives of the coordinates with respect to $a, e, \Omega, i, \pi, \varepsilon$. In fact equations (2) become identities when for the coordinates their functions of t and the six constants are substituted. In this sense, the differentiation of (1) with respect to a gives

$$(3) \quad * \quad \frac{\partial^2\frac{\partial x}{\partial a}}{\partial t^2} = -\frac{\mu\frac{\partial x}{\partial a}}{r^3} + 3\mu\frac{x}{r^5}\left(x\frac{\partial x}{\partial a} + y\frac{\partial y}{\partial a} + z\frac{\partial z}{\partial a}\right),$$

and corresponding equations hold for the derivatives with respect to the other constants. Therefore, on the assumption that $X = Y = Z = 0$, the complete solution of (1) takes the form

$$(4) \quad * \quad \delta x = A\frac{\partial x}{\partial a} + B\frac{\partial x}{\partial e} + C\frac{\partial x}{\partial \Omega} + D\frac{\partial x}{\partial i} + E\frac{\partial x}{\partial \pi} + F\frac{\partial x}{\partial \varepsilon},$$

in which A, B, C, D, E, F are six arbitrary constants.

According to Lagrange's theory of the variation of constants, the solution of equations (1) have the forms (4) even when X, Y, Z are not zero, but are any given functions of t, only in this case $A, B, \ldots$ are no longer constants but functions of t which remain to be more exactly determined. The sub-

stitution of (4) in (1) gives only three equations between the six variables $A, B, \ldots$ and this enables three other equations between them to be selected arbitrarily. Conformably to this theory these are

$$(5) \qquad * \qquad 0 = \frac{dA}{dt} \cdot \frac{\partial x}{\partial a} + \frac{dB}{dt} \cdot \frac{\partial x}{\partial e} + \ldots$$

and by means of them, equations (4), by differentiation, take the form

$$(6) \qquad * \qquad \frac{d\delta x}{dt} = A \cdot \frac{d\frac{\partial x}{\partial a}}{dt} + B \cdot \frac{d\frac{dx}{de}}{dt} + \ldots$$

as if $A, B, \ldots$ were constant.

The differentiation of (6) gives

$$(7) \qquad * \qquad \frac{d^2\delta x}{dt^2} = A \cdot \frac{d^2\frac{\partial x}{\partial a}}{dt^2} + B \cdot \frac{d^2\frac{\partial x}{\partial e}}{dt^2} + \ldots$$
$$+ \frac{dA}{dt} \cdot \frac{d\frac{\partial x}{\partial a}}{dt} + \frac{dB}{dt} \cdot \frac{d\frac{\partial x}{\partial e}}{dt} + \ldots,$$

and by substituting (7) and (4) and using (3), the equations (1) take the forms

$$(8) \qquad \begin{aligned} X &= \frac{dA}{dt}\frac{\partial u}{\partial a} + \frac{dB}{dt}\frac{du}{de} + \ldots \\ Y &= \frac{dA}{dt}\frac{dv}{\partial a} + \frac{dB}{dt}\frac{\partial v}{\partial e} + \ldots \\ Z &= \frac{dA}{dt}\frac{\partial w}{\partial a} + \frac{dB}{dt}\frac{\partial w}{de} + \ldots \end{aligned}$$

in which u, v, w are, as before, the component velocities. Equations (5) and (8) are the six equations which determine the variables $A, B, \ldots$. If they are solved for $\frac{dA}{dt}, \frac{dB}{dt}, \ldots$, they immediately give by the use of (60), § 10, developments of the form

$$(9) \qquad \begin{cases} \dfrac{dA}{dt} = X\dfrac{\partial a}{\partial u} + Y\dfrac{\partial a}{\partial v} + Z\dfrac{\partial a}{\partial w}, \\ \dfrac{dB}{dt} = X\dfrac{\partial e}{\partial u} + Y\dfrac{\partial e}{\partial v} + Z\dfrac{\partial e}{\partial w}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{cases}$$

These equations can be at once integrated and give $A, B, \ldots,$ as functions of t, which, when used in (4) furnish the perturbations $\delta x, \delta y, \delta z$.

This completes the solution so far as the integration of the equations (1) is concerned. But, on account of the developments which follow, equations (9) will be transformed by the employment of the principles developed in § 10. In this case, the equations (58) by the use of (62) and (63) become

$$(10)\quad \begin{cases} \dfrac{\partial a}{\partial x} = (a,a)\dfrac{\partial u}{\partial a} + (a,e)\dfrac{\partial u}{\partial e} + (a,\Omega)\dfrac{\partial u}{\partial \Omega} + \ldots \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{cases}$$

$$(11)\quad \begin{cases} \dfrac{\partial a}{\partial u} = -(a,a)\dfrac{\partial x}{\partial a} - (a,e)\dfrac{\partial x}{\partial e} - (a,\Omega)\dfrac{\partial x}{\partial \Omega} - \ldots \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{cases}$$

By the substitution of (10) and the corresponding equations in (9), and for brevity putting,

$$(12)\quad \begin{cases} \dfrac{\partial R}{\partial a} = X\dfrac{\partial x}{\partial a} + Y\dfrac{\partial y}{\partial a} + Z\dfrac{\partial z}{\partial a}, \\ \dfrac{\partial R}{\partial e} = X\dfrac{\partial x}{\partial e} + Y\dfrac{\partial y}{\partial e} + Z\dfrac{\partial z}{\partial e}, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{cases}$$

we obtain the system

$$(13)\quad \begin{cases} \dfrac{dA}{dt} = (a,a)\dfrac{\partial R}{\partial a} + (e,a)\dfrac{\partial R}{\partial e} + (\Omega,a)\dfrac{\partial R}{\partial \Omega} + \ldots \\ \dfrac{dB}{dt} = (a,e)\dfrac{\partial R}{\partial a} + (e,e)\dfrac{\partial R}{\partial e} + (\Omega,e)\dfrac{\partial R}{\partial \Omega} + \ldots \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{cases}$$

and, since Poisson's expressions are independent of the time, it at once follows from (13), that

$$(14)\quad \begin{cases} A = (a,a)\displaystyle\int^t \dfrac{\partial R}{\partial a}dt + (e,a)\int^t \dfrac{\partial R}{\partial e}dt + \ldots \\ B = (a,e)\displaystyle\int^t \dfrac{\partial R}{\partial a}dt + (e,e)\int^t \dfrac{\partial R}{\partial e}dt + \ldots \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{cases}$$

The perturbations δx, δy, δz receive, by another use of (10), the forms

$$(15)\quad \begin{cases} \delta x = \dfrac{\partial a}{\partial u}\displaystyle\int^t \frac{\partial R}{\partial a}dt + \frac{\partial e}{\partial u}\int^t \frac{\partial R}{\partial e}dt + \dots \\ \delta y = \dfrac{\partial a}{\partial v}\displaystyle\int^t \frac{\partial R}{\partial a}dt + \frac{\partial e}{\partial v}\int^t \frac{\partial R}{\partial e}dt + \dots \\ \delta z = \dfrac{\partial a}{\partial w}\displaystyle\int^t \frac{\partial R}{\partial a}dt + \frac{\partial e}{\partial w}\int^t \frac{\partial R}{\partial e}dt + \dots \end{cases}$$

and, finally the derivatives of the perturbations with respect to the time follow from (6), and being denoted by δu, δv, δw, are

$$(16)\quad \begin{cases} \delta u = -\dfrac{\partial a}{\partial x}\displaystyle\int^t \frac{\partial R}{\partial a}dt - \frac{\partial e}{\partial x}\int^t \frac{\partial R}{\partial e}dt - \dots \\ \delta v = -\dfrac{\partial a}{\partial y}\displaystyle\int^t \frac{\partial R}{\partial a}dt - \frac{\partial e}{\partial y}\int^t \frac{\partial R}{\partial e}dt - \dots \\ \delta w = -\dfrac{\partial a}{\partial z}\displaystyle\int^t \frac{\partial R}{\partial a}dt - \frac{\partial e}{\partial z}\int^t \frac{\partial R}{\partial e}dt - \dots \end{cases}$$

It is worthy of notice that in each of the six integrals the lower limit is arbitrary, so that six new arbitrary constants appear in (15) and (16). This is easy to understand for we have here the solutions of the systems (1) of the sixth order. In passing to the perturbations of higher and higher orders, according to the preceding section, the system must be solved again and again, and we thus reach the series (7) of this section which, by hypothesis, is convergent.

A circumstance here appears, which, at first sight, is very surprising, for in this development we get not only more arbitrary constants than we need but arbitrary constants without number. This has given rise to much controversy. The first approximation, Kepler's ellipse, affords the necessary number of constants. So it appeard that no new ones should be introduced in the perturbations, though this was done by Laplace

in the first volume of *Mécanique Célèste.* The legitimacy of an unlimited number of arbitrary constants of integration can be shown as follows:

If a system of n differential equations between $(n+1)$ quantities $x_1, \dots x_{n+1}$ be given, the integration will give n of these $x_1, \dots x_n$ in terms of the last x_{n+1}, and a sufficient number of arbitrary constants $a_1, \dots a_\lambda$. If now the differential equations contain a certain number of parameters $p_1, \dots p_\mu$, these will appear in the solutions, and $x_1, \dots x_n$ will be functions of x_{n+1} and also of $a_1, \dots a_\lambda$ and of $p_1, \dots p_\mu$. But, since the only property of $a_1, \dots a_\lambda$ is their being constants, they may be replaced by arbitrary functions of $p_1, \dots p_\mu$, and as we can introduce into these functions an unlimited number of arbitrary constants, the solution can be so taken as to include any number of them.

A system of total differential equations passes at once into a system of partial differential equations, if certain parameters occur in them and if attention is directed to the manner in which these parameters enter the solution. If, for example, the differential equation $f\left(x, y, \frac{dy}{dx}, p\right)$ is given, and the function $y = \varphi(x, p)$ sought, then x and p are to be regarded as the two independent variables and the differential equation, a partial one of the first order; in it, however, of the two partial derivatives $\frac{\partial y}{\partial x}, \frac{\partial y}{\partial p}$, only the first appears.

Consequently, it appears that the unlimited number of constants is due to the nature of the method; they must finally be reduced to the necessary number. In the theory of absolute perturbations, the only limitation is that the series must converge.

In order to make the method definite in its application, a special law must be adopted to remove the indefiniteness in the selection of the constants. The following is such a law: *All perturbations and their derivatives with respect to the time must vanish for a definite instant* t_0. Equations (15) and (16) show that this will be the case when t_0 is made the lower limit of integration. The forms of the definite integrals then become

$$\int_{t_0}^{t} \frac{\partial R}{\partial a}\,dt, \quad \int_{t_0}^{t} \frac{\partial R}{\partial e}\,dt, \ldots$$

The form under which the perturbations now appear are called *special*, in so far as they are reckoned from a definite instant t_0. These special perturbations were included in the ideas of the founders of the theory. If, for any instant, the coordinates and velocities are given the elements $a, e, \ldots$ are thereby determined. An ellipse would thus be obtained for each planet and the planet would move in this ellipse were it not for the perturbing functions. They constantly cause the planet to depart from an elliptic path and this departure is the perturbation. In the general theory of perturbations, the original fundamental ellipse may be such that the planet at no moment moves in it, although on account of the convergence, an ellipse may be selected which, during the interval of time included in the integration, very nearly represents the actual path of the planet.

So far as is known to the author, Jacobi was the first who insisted clearly and without ambiguity on the appearance of an unlimited number of constants of integration in the theory of absolute perturbations. (Extracts from two letters to Hansen, *Crelle's Journal*, Bd. 42).

As was seen in the beginning, the theory of absolute perturbations rests on the assumption that the series are convergent. The question now aries: Up to what time and with what selection of constants of integration is this true? The investigations required for the answer of this question have not yet been made. Formerly but little attention was paid to considerations of convergency, but now they play an important part in every investigation of series, and on this account the absence of a proof of convergency, or rather as I prefer to say, the failure of the proofs of convergency in this case, is a serious defect. This, however, can hardly be made a series of reflections on the talented founders of the theories. On the contrary, they deserve high credit for having intuitively adopted a legitimate process, the demonstration of which was left to later times.

While there is no rigorous test of the convergency, yet it is highly probable for tolerably long periods. For instance, in the case of special perturbations, the values of the perturbations and of their first derivatives are $=0$ for the time t_0. Those of the n^{th} degree are homogeneous functions of the n^{th} degree in the disturbing masses, and as these masses are small it is highly probable that the perturbations decrease as the degree increases.

Since the integrals extend from t_0 to t, it follows that the perturbations will generally grow larger as t increases, and it is probable, though not absolutely certain, that the convergence will finally cease. It is therefore necessary to divide a long period into parts and to consider each part by itself.

The advantage of absolute perturbations consists in the smallness of the disturbing masses in terms of the powers and products of which the perturbations are expressed. They will, therefore, as is shown by actual application, converge much more rapidly and through a longer period than will be the case with those depending on series developed in ascending powers of t and in which the advantage afforded by the smallness of the masses is lost. In practice, the interval of time is taken so small that the perturbations of the first degree, with the leading term of the second, usually afford a sufficient approximation.

Finally, since the perturbations of the first degree are by far the most important, we will reduce them to a very simple and elegant analytical form. From § 21 it follows that the quantities X, Y, Z are given without further development, and that they become $\dfrac{\partial R}{\partial x}, \dfrac{\partial R}{\partial y}, \dfrac{\partial R}{\partial z}$, if for the coordinates are substituted the first terms of the second member of (8), that is, the so-called elliptic values. The quantity R introduced in § 21 is, therefore, really present, and in fact is the perturbing function when the elliptic values are substituted for the coordinates in it. The integrals which appear in (15) and (16) are now to be calculated by expressing R in terms of the elements $a, e, \ldots$ and t, then differentiating with respect to the elements and integrating with respect to t between the limits t_0 and t. The same lower limit t_0 will be selected for all integrals, therefore special per-

turbations are to be used. We can, however, according to the rule for differentiation under the sign of integration, first integrate with respect to t and then differentiate with respect to the elements $a, e, \ldots.$ If we put

$$\int_{t_0}^{t} R_\lambda dt = U_\lambda, \tag{17}$$

the equations (15) and (16) become

$$\left\{\begin{aligned}
\delta x_1 &= \frac{\partial U_1}{\partial a_1}\frac{\partial a_1}{\partial u_1} + \frac{\partial U_1}{\partial e_1}\frac{\partial e_1}{\partial u_1} + \ldots \\
\delta y_1 &= \frac{\partial U_1}{\partial a_1}\frac{\partial a_1}{\partial v_1} + \frac{\partial U_1}{\partial e_1}\frac{\partial e_1}{\partial v_1} + \ldots \\
\delta z_1 &= \frac{\partial U_1}{\partial a_1}\frac{\partial a_1}{\partial w_1} + \frac{\partial U_1}{\partial e_1}\frac{\partial e_1}{\partial w_1} + \ldots \\
\delta u_1 &= -\frac{\partial U_1}{\partial a_1}\frac{\partial a_1}{\partial x_1} + \frac{\partial U_1}{\partial e_1}\frac{\partial e_1}{\partial x_1} + \ldots \\
\delta v_1 &= -\frac{\partial U_1}{\partial a_1}\frac{\partial a_1}{\partial y_1} + \frac{\partial U_1}{\partial e_1}\frac{\partial e_1}{\partial y_1} + \ldots \\
\delta w_1 &= -\frac{\partial U_1}{\partial a_1}\frac{\partial a_1}{\partial z_1} + \frac{\partial U_1}{\partial e_1}\frac{\partial e_1}{\partial z_1} + \ldots \\
&\ldots \qquad \ldots\ldots \\
&\ldots \qquad \ldots\ldots
\end{aligned}\right. \tag{18}$$

These formulas may be made more compact. The U's are, as appears by the integration of (17), given functions of $t, t_0, a_1, e_1, \ldots a_2, e_2, \ldots$ If we consider the elements replaced by their (elliptic) coordinates and component velocities, the equations (18) become, simply,

$$\left\{\begin{aligned}
x_\lambda &= \frac{\partial U_\lambda}{\partial u_\lambda}, & \delta y_\lambda &= \frac{\partial U_\lambda}{\partial v_\lambda}, & z_\lambda &= \frac{\partial U_\lambda}{\partial w_\lambda}, \\
\delta u_\lambda &= -\frac{\partial U_\lambda}{\partial x_\lambda}, & \delta v_\lambda &= -\frac{\partial U_\lambda}{\partial y_\lambda}, & \delta w_\lambda &= -\frac{\partial U_1}{\partial z_1}
\end{aligned}\right. \tag{19}$$

where $\lambda = 1, \ldots n.$

These equations give the solution, therefore, in the following very simple analytical form:

For the coordinates in the perturbing function, substitute their elliptic values expressed in terms of the elements and the time and integrate between the limits t_0 and t. In the integral re-express the elements, at least for those planets to which the perturbing function belongs, in terms of the coordinates and component velocities. The perturbations of the coordinates are then the derivatives of the integral with respect to the component velocities. Also the derivatives of the perturbations of the coordinates, that is the perturbations of the component velocities, are the negative derivatives of this integral with respect to the coordinates.

The perturbations of a planet are thus reduced to the integral (17) which, for this reason, is called the integral of the perturbing function.

23. Other Formulas for the Absolute Perturbations.

While the formulas of the preceding section, especially (19), for the perturbations of the first degree, are of great elegance, they are not so suitable for actual computation as those given by Laplace in *Mécanique célèste*, Tome I, p. 281. We will now develop these formulas.

If the original plane of the orbit be taken as the plane of reference, the (elliptic) $z=0$ and the last of equations (1) in the preceding section, become

$$\frac{d^2\delta z}{dt^2} = -\mu\frac{\delta z}{r^3} + Z. \tag{1}$$

For $Z=0$, the general solution of (1) is

$$\delta z = Ax + By. \tag{2}$$

This form will also hold for δz when Z is any given function of t, in which case, however, A and B are functions of t which are to be determined. As a differential equation between them we select

$$0 = \frac{dA}{dt}x + \frac{dB}{dt}y, \tag{3}$$

so that we get by the differentation of (2)

(4) $$\frac{d\delta z}{dt} = A\frac{dx}{dt} + B\frac{dy}{dt},$$

and

(5) $$\frac{d^2\delta z}{dt^2} = A\frac{d^2x}{dt^2} + B\frac{d^2y}{dt^2} + \frac{dA}{dt}\frac{dx}{dt} + \frac{dB}{dt}\frac{dy}{dt}$$

$$= -A\mu\frac{x}{r^3} - B\mu\frac{y}{r^3} + \frac{dA}{dt}\frac{dx}{dt} + \frac{dB}{dt}\frac{dy}{dt}.$$

Finally by substituting (2) and (5) in (1), we get

(6) $$\frac{dA}{dt}\frac{dx}{dt} + \frac{dB}{dt}\frac{dy}{dt} = Z,$$

and from (3) and (6)

(7) $$\begin{cases} \dfrac{dA}{dt} = -\dfrac{yZ}{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}} = -\dfrac{yZ}{\sqrt{\mu a(1-e^2)}}, \\[2ex] \dfrac{dB}{dt} = \dfrac{xZ}{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}} = \dfrac{xZ}{\sqrt{\mu a(1-e^2)}}, \end{cases}$$

and, therefore, if we limit the perturbations to the first degree, for which $Z = \dfrac{\partial R}{\partial z}$,

(8) $$A = -\frac{\int^t y\frac{\partial R}{\partial z}dt}{\sqrt{\mu a(1-e^2)}}, \qquad B = \frac{\int^t x\frac{\partial R}{\partial z}dt}{\sqrt{\mu a(1-e^2)}},$$

and, thereby we get

(9) $$\begin{cases} \delta z = \dfrac{y\int^t x\dfrac{\partial R}{\partial z}dt - x\int^t y\dfrac{\partial R}{\partial z}dt}{\sqrt{\mu a(1-e^2)}}, \\[3ex] \dfrac{d\delta z}{dt} = \dfrac{\dfrac{dy}{dt}\int^t x\dfrac{\partial R}{\partial z}dt - \dfrac{dx}{dt}\int^t y\dfrac{\partial R}{\partial z}dt}{\sqrt{\mu a(1-e^2)}}. \end{cases}$$

To prepare for further integration of (1), it is desirable to consider the expression

(10) $$x\delta x + y\delta y = P.$$

By two differentiations of P, we obtain by the use of the equation (1) of the preceding section, in which z is to put equal to zero,

$$(11)\qquad \frac{d^2P}{dt^2} = \frac{\mu P}{r^3} + 2\left(\frac{dx}{dt}\frac{d\delta x}{dt} + \frac{dy}{dt}\frac{d\delta y}{dt}\right) + Xx + Yy.$$

If we put

$$(12)\qquad \frac{dx}{dt}\frac{d\delta x}{dt} + \frac{dy}{dt}\frac{d\delta t}{dt} = Q,$$

then

$$\frac{dQ}{dt} = \frac{dx}{dt}\frac{d^2\delta x}{dt^2} + \frac{dy}{dt}\frac{d^2\delta y}{dt^2} + \frac{d^2x}{dt^2}\frac{d\delta x}{dt} + \frac{d^2y}{dt^2}\frac{d\delta y}{dt}$$

$$= -\mu \, . \, \frac{\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y\right)}{r^3} + 3\mu \, . \, \frac{r\frac{dr}{dt}}{r^5} \, . \, P$$

$$-\mu \, . \, \frac{x\frac{d\delta x}{dt} + y\frac{d\delta y}{dt}}{r^3} + X\frac{dx}{dt} + Y\frac{dy}{dt}$$

$$(13)\qquad = \frac{d\left(-\frac{\mu P}{r^3}\right)}{dt} + X\frac{dx}{dt} + Y\frac{dy}{dt},$$

and by direct integration

$$(14)\qquad Q = -\frac{\mu P}{r^3} + \int^t (Xdx + Ydy).$$

Equation (11) now becomes

$$(15)\qquad \frac{d^2P}{dt^2} = -\frac{\mu P}{r^3} + Xx + Yy + 2\int^t (Xdx + Ydy).$$

Now

$$x = a\frac{\partial x}{\partial a}, \quad y = a\frac{\partial y}{\partial a}$$

and therefore

$$Xx + Yy = a\left(\frac{\partial R}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial a}\right) = a\frac{\partial R}{\partial a},$$

$$X\frac{dx}{dt} + Y\frac{dy}{dt} = n\left(\frac{\partial R}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial \varepsilon}\right) = n\frac{\partial R}{\partial \varepsilon}.$$

For brevity, putting

(16) $$W = a\frac{\partial R}{\partial a} + 2n\int \frac{\partial R}{\partial \varepsilon}dt,$$

equation (15) becomes

(17) $$\frac{d^2P}{dt^2} = -\frac{\mu P}{r^3} + W.$$

This equation is of the same form as (1) and is to be integrated in the same manner. Thus we get

(18) $$\begin{cases} P = \dfrac{y\int^t xWdt - x\int^t yWdt}{\sqrt{\mu a(1-e^2)}}, \\[2ex] \dfrac{dP}{dt} = \dfrac{\dfrac{dy}{dt}\int^t x\,Wdt - \dfrac{dx}{dt}\int^t y\,Wdt}{\sqrt{\mu a(1-e^2)}}. \end{cases}$$

These express the relation between δx and δy and equation (10). In order to obtain another, consider the expression

(19) $$S = \frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y.$$

By differentation

$$\frac{dS}{dt} = -\frac{\mu P}{r^3} + Q,$$

and therefore by (17) and (14),

$$\frac{dS}{dt} = 2\frac{d^2P}{dt^2} - 2a\frac{\partial R}{\partial a} - 3n\int^t \frac{\partial R}{\partial \varepsilon}dt,$$

and by integration

(20) $$S = 2\frac{dP}{dt} - 2a\int^t \frac{\partial R}{\partial a}dt - 3n\int^t\int^t \frac{\partial R}{\partial \varepsilon}dt\,dt,$$

in which for $\dfrac{dP}{dt}$ is to be substituted its value from (18). Finally, after the determination of P and S, the perturbations follow

(21) $$\begin{cases} \delta x = \dfrac{P\dfrac{dy}{dt} - Sy}{\sqrt{\mu a(1-e^2)}}, \\[2ex] \delta y = \dfrac{-P\dfrac{dx}{dt} + Sx}{\sqrt{\mu a(1-e^2)}}. \end{cases}$$

The perturbations are thereby obtained. By the introduction of the polar coordinates r, v, φ, by the equations

$$x = r\cos v\cos\varphi,$$
$$y = r\sin v\cos\varphi,$$
$$z = r\sin\varphi;$$

we obtain, since $\varphi = 0$ when we limit the perturbations to the first degree,

$$\delta x = \cos v\,\delta r - r\sin v\,\delta v,$$
$$\delta y = \sin v\,\delta r + r\cos v\,\delta v,$$
$$\delta z = r\,\delta\varphi,$$

and therefore

$$\delta r = \cos v\,\delta x + \sin v\,\delta y = \frac{P}{r}$$

$$\delta v = \frac{-\sin v\,\delta x + \cos v\,\delta y}{r} = \frac{-y\delta x + x\delta y}{r} = \frac{-P\frac{rdr}{dt} + Sr^2}{r^2\sqrt{\mu a(1-e^2)}},$$

and therefore

$$(22)\quad \begin{cases} \delta r = \dfrac{\sin v\displaystyle\int^t r\cos v\,Wdt - \cos v\displaystyle\int^t r\sin v\,Wdt}{\sqrt{\mu a(1-e^2)}}, \\[2ex] \delta v = \dfrac{-\dfrac{dr}{dt}\delta r + 2\dfrac{d(r\delta r)}{dt} - 2\displaystyle\int^t a\frac{\partial R}{\partial a}dt - 3\int^t\!\!\int^t \left(n\frac{\partial R}{\partial \varepsilon}dt\right)dt}{\sqrt{\mu a(1-e^2)}}, \\[2ex] \delta\varphi = \dfrac{\sin v\displaystyle\int^t r\cos v\frac{\partial R}{\partial z}dt - \cos v\int^t r\sin v\frac{\partial R}{\partial z}dt}{\sqrt{\mu a(1-e^2)}}. \end{cases}$$

Of these quantities, r is called the radius vector of the planet, v its longitude, and φ its latitude. Equations (22) furnish the final expressions for the perturbations of these coordinates.

The two methods of calculating the perturbations, given in this and the preceding section, give results which appear totally different. For instance, in equations (15), § 22, there are only simple integrals; in (22), § 23, $\frac{\partial R}{\partial \varepsilon}$ requires a double integration.

Besides this the two sets of formulas are so different that it is not possible to pass directly from one to the other. We will now consider the cause of this difference and the method to be taken to prove the identity of the two systems.

In the first place it is to be noted that in (15), § 22, the entire expression $\frac{\partial R}{\partial a}$ is to be taken and that a appears in R not alone explicitly but also in the combination $\zeta = \sqrt{\frac{\mu}{a^3}}t + \varepsilon$. If for better distinction, the total derivatives are designated by parentheses, we have

$$\left(\frac{\partial R}{\partial a}\right) = \frac{\partial R}{\partial a} - \frac{3}{2}\frac{n}{a} \cdot t \cdot \frac{\partial R}{\partial \varepsilon},$$

and, therefore,

$$\int\left(\frac{\partial R}{\partial a}\right)dt = \int\frac{\partial R}{\partial a}dt - \frac{3n}{2a}\int t\frac{\partial R}{\partial \varepsilon}dt.$$

Now

$$\int t\frac{\partial R}{\partial \varepsilon}dt = t\int\frac{\partial R}{\partial \varepsilon}dt - \int\left(\int\frac{\partial R}{\partial \varepsilon}dt\right)dt,$$

and we see that double integration may also be introduced in (15) of the preceding section. Yet even in this case, we can not transform (15), § 22 into (22), § 23. The two systems are connected by three partial differential equations which are satisfied by R taken as a function of the elements, $a, e, \Omega, \pi, \varepsilon$. Originally, R is a function of the coordinates, the velocities not appearing in it. If we consider R expressed as a function of t and the elements, and then substitute the coordinates and velocities for them, the last must vanish. In this sense, it follows, therefore, that

$$(23)\qquad \begin{cases} 0 = \dfrac{\partial R}{\partial u} = \dfrac{\partial R}{\partial a}\dfrac{\partial a}{\partial u} + \dfrac{\partial R}{\partial e}\dfrac{\partial e}{\partial u} + \dots \\ 0 = \dfrac{\partial R}{\partial v} = \dfrac{\partial R}{\partial a}\dfrac{\partial a}{\partial v} + \dfrac{\partial R}{\partial e}\dfrac{\partial e}{\partial v} + \dots \\ 0 = \dfrac{\partial R}{\partial w} = \dfrac{\partial R}{\partial a}\dfrac{\partial a}{\partial w} + \dfrac{\partial R}{\partial e}\dfrac{\partial e}{\partial w} + \dots \end{cases}$$

In these expressions we have to regard the values of

$\frac{\partial a}{\partial u}, \frac{\partial e}{\partial u}, \ldots$ in terms of the elements and the time to be substituted for them in order to determine the three partial differential equations to which reference has been made.

With the help of (23), the identity of the two sets of equations for the perturbations can be shown. Moreover, it is clear, that we can use the equations (23), to represent the perturbations in a great number of forms and ways.

If the formulas (22) are to be used in any special case, there are two different ways of proceeding. The integral may, first, be evaluated by taking X, Y, Z, and the coordinates at suitable and equal intervals of time, (perhaps every ten days), and then using a mechanical quadrature. This process disregards all analytical considerations and gives only numerical results. It is suitable especially for comets and asteriods, for which, on account of their great inclinations and eccentricities, the analytical developments are too voluminous.

The other process consists in determining the analytical expressions for the perturbations. These depend upon the perturbing function which separates into two parts, of which each contains the coordinates of only two planets, the disturbing and the disturbed. Since the equations (15), §22, contain only the derivatives of the perturbing function with respect to the elements, the question resolves itself into the investigation of the perturbing function as a function of the elements and the time. This investigation will be the subject of the following sections.

24. Analytical Development of the Perturbing Function.

The analytical development of the perturbing function has engaged the attention of many astronomers and mathematicians. The process adopted here is that used by Leverrier in his *Recherches Astronomiques.* According to (3) §21, the perturbing function R consists of parts which depend only upon the coordinates of two planets and which appear in two different forms. The first of these forms is

(1) $$r_{12}^{-1}=\frac{1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}},$$

and the second is

(2) $$Q=\frac{x_1x_2+y_1y_2+z_1z_2}{(\sqrt{x_2^2+y_2^2+z_2^2})^3}.$$

In what follows, we propose to develop these two functions in ascending powers of eccentricities and inclinations, since these elements are small. We will first transform them somewhat.

If in (23), § 2, we put for brevity,

(3) $$v+\pi-\Omega=\omega,$$

we get

(4) $$\begin{cases} x=r[\cos(\Omega+\omega)\cos^2\frac{1}{2}i+\cos(\Omega-\omega)\sin^2\frac{1}{2}i], \\ y=r[\sin(\Omega+\omega)\cos^2\frac{1}{2}i+\sin(\Omega-\omega)\sin^2\frac{1}{2}i], \\ z=r\sin\omega\sin i, \end{cases}$$

and therefore,

$$\begin{aligned} x_1x_2+y_1y_2+z_1z_2=r_1r_2[&\cos^2\tfrac{1}{2}i_1\cos^2\tfrac{1}{2}i_2\cos(\Omega_1-\Omega_2+\omega_1-\omega_2) \\ &+\cos^2\tfrac{1}{2}i_1\sin^2\tfrac{1}{2}i_2\cos(\Omega_1-\Omega_2+\omega_1+\omega_2) \\ &+\sin^2\tfrac{1}{2}i_1\cos^2\tfrac{1}{2}i_2\cos(\Omega_2-\Omega_1+\omega_1+\omega_2) \\ &+\sin^2\tfrac{1}{2}i_1\sin^2\tfrac{1}{2}i_2\cos(\Omega_1-\Omega_2-\omega_1+\omega_2) \\ &+\tfrac{1}{2}\sin i_1\sin i_2\cos(\omega_1-\omega_2) \\ &-\tfrac{1}{2}\sin i_1\sin i_2\cos(\omega_1+\omega_2)], \end{aligned}$$

or,

$$\begin{aligned} 2(x_1x_2+y_1y_2+z_1z_2)=r_1r_2[&m\cos(\omega_1-\omega_2)+n\sin(\omega_1-\omega_2) \\ &+o\cos(\omega_1+\omega_2)+p\sin(\omega_1+\omega_2)], \end{aligned}$$

in which

$$\begin{aligned} m&=(1+\cos i_1\cos i_2)\cos(\Omega_1-\Omega_2)+\sin i_1\sin i_2, \\ n&=-(\cos i_1+\cos i_2)\sin(\Omega_1-\Omega_2), \\ o&=(1-\cos i_1\cos i_2)\cos(\Omega_1-\Omega_2)-\sin i_1\sin i_2, \\ p&=-(\cos i_1-\cos i_2)\sin(\Omega_1-\Omega_2). \end{aligned}$$

The four quantities m, n, o, p depend upon the three angle i_1, i_2, $\Omega_1-\Omega_2$. A relation must, therefore, exist between them. If we introduce the angle between the planes of the orbits J, we have

(5) $$\cos J=\cos i_1\cos i_2+\sin i_1\sin i_2\cos(\Omega_1-\Omega_2),$$

and an easy reduction gives

$$m^2+n^2=(1+\cos J)^2=4\cos^4\tfrac{1}{2}J,$$
$$o^2+p^2=(1-\cos J)^2=4\sin^4\tfrac{1}{2}J,$$

consequently,

$$\sqrt{m^2+n^2}+\sqrt{o^2+p^2}=2.$$

Let Π_1 and Π_2 be the angles between the nodes of the planes of the orbits on the xy plane and the intersection of the planes of the orbits, and we have the equations,

$$(6)\quad\left\{\begin{aligned}
\cos(\Pi_1-\Pi_2)&=\frac{m}{\sqrt{m^2+n^2}}\\
&=\frac{(1+\cos i_1\cos i_2)\cos(\Omega_1-\Omega_2)+\sin i_1\sin i_2}{1+\cos J},\\
\sin(\Pi_1-\Pi_2)&=-\frac{n}{\sqrt{m^2+n^2}}=\frac{(\cos i_1+\cos i_2)\sin(\Omega_1-\Omega_2)}{1+\cos J},\\
\cos(\Pi_1+\Pi_2)&=\frac{o}{\sqrt{o^2+p^2}}\\
&=\frac{(1-\cos i_1\cos i_2)\cos(\Omega_1-\Omega_2)-\sin i_1\sin i_2}{1-\cos J},\\
\sin(\Pi_1+\Pi_2)&=-\frac{p}{\sqrt{o^2+p^2}}=\frac{(\cos i_1-\cos i_2)\sin(\Omega_1-\Omega_2)}{1-\cos J},
\end{aligned}\right.$$

and finally,

$$(7)\quad x_1x_2+y_1y_2+z_1z_2=r_1r_2[\cos(\omega_1+\Pi_1-(\omega_2+\Pi_2)]\cos^2\tfrac{1}{2}J$$
$$+\cos(\omega_1+\Pi_1+\omega_2+\Pi_2)\sin^2\tfrac{1}{2}J],$$

and, therefore, if for brevity, we put

$$(8)\qquad \omega_1+\Pi_1-(\omega_2+\Pi_2)=V,$$

$$(9)\qquad \omega_1+\Pi_1+(\omega_2+\Pi_2)=W,$$

we get,

$$(10)\quad r_{12}^{-1}=[r_1^2+r_2^2-2r_1r_2(\cos V\cos^2\tfrac{1}{2}J+\cos W\sin^2\tfrac{1}{2}J)]^{-\frac{1}{2}}.$$

The perturbing function r_{12}^{-1}, is, therefore, expressed in terms of the five quantities

$$r_1,\ r_2,\ V,\ W,\ J,$$

and it has the form

$$(11)\qquad r_{12}^{-1} = (a + b\cos V + c\cos W)^{-\frac{1}{2}},$$

in which

$$a = r_1^2 + r_2^2,$$
$$b = -2r_1 r_2 \cos^2 \tfrac{1}{2} J,$$
$$c = -2r_1 r_2 \sin^2 \tfrac{1}{2} J.$$

If, as we shall assume, r_1 is always greater than r_2, or r_2 always greater than r_1, then since a is greater than the absolute values of b and c, we can assume the series

$$(12)\qquad r_{12}^{-1} = \Sigma q_{\alpha,\beta} \cos(\alpha V + \beta W),$$

in which α and β take all possible positive and negative integral values and the q's denote the coefficients which are to be arranged in ascending powers of b and c. These coefficients possess very interesting properties, but they cannot be discussed here. The reader is referred for them to Jacobi's paper in *Crelle's Journal*, Bd. 15, entitled, *De evolutione expressionis* $(l + 2l' \cos\varphi + 2l'' \cos\varphi')^{-n}$ *in seriem infinitam secundum cosinus multiplorum utriusque anguli* φ, φ' *procedentem.*

We shall use the circumstance that

$$(13)\qquad z = \sin^2 \tfrac{1}{2} J$$

is a small quantity and develop (10) in ascending powers of it. If, for brevity, we put

$$(14)\qquad \rho = r_1^2 + r_2^2 - 2r_1 r_2 \cos V,$$

then

$$(15)\qquad \begin{aligned} r_{12}^{-1} &= [\rho + 2r_1 r_2(\cos V - \cos W)z]^{-\frac{1}{2}} \\ &= \rho^{-\frac{1}{2}} - r_1 r_2(\cos V - \cos W)z\rho^{-\frac{3}{2}} \\ &\qquad + \tfrac{3}{2}[r_1 r_2(\cos V - \cos W)z]^2 \rho^{-\frac{5}{2}} \\ &\qquad - \tfrac{5}{2}[r_1 r_2(\cos V - \cos W)z]^3 \rho^{-\frac{7}{2}} \\ &\qquad \cdots\cdots\cdots \end{aligned}$$

On account of the smallness of z, only a few terms of this development are needed. To complete the form (12), the powers of $(\cos V - \cos W)$ may be arranged in terms of the cosines of the multiple angles. This gives

$$
(16)\quad \begin{cases} 2(\cos V-\cos W)^2 = 2+\cos 2V+\cos 2W-2\cos(V-W) \\ \qquad\qquad\qquad\qquad\qquad\qquad -2\cos(V+W), \\ 4(\cos V-\cos W)^3 = 9\cos V-9\cos W+\cos 3V-\cos 3W \\ \qquad\qquad -3\cos(2V+W)+3\cos(V+2W) \\ \qquad\qquad -3\cos(2V-W)+3\cos(V-2W). \end{cases}
$$

Further, let

$$
(17)\qquad \rho^{-\frac{n}{2}} = (r_1^2+r_2^2-2r_1r_2\cos V)^{-\frac{n}{2}} = \tfrac{1}{2}\sum_{\lambda=-\infty}^{\lambda=+\infty}(n^\lambda)\cos(\lambda V).
$$

This formula, in which it is assumed that

$$
(18)\qquad (n^\lambda) = (n^{-\lambda}),
$$

is fundamental in the analytical development of the perturbing function, and on this account we shall treat it in detail.

If φ is any selected angle, then it appears that

$$
\cos\varphi\,.\,\tfrac{1}{2}\Sigma(n^\lambda)\cos\lambda V = \tfrac{1}{4}\Sigma(n^\lambda)(\cos\lambda V+\varphi)+\tfrac{1}{4}\Sigma(n^\lambda)\cos(-\lambda V+\varphi).
$$

The two terms in the second member are alike as may be seen from (18) by exchanging λ and $-\lambda$. By the use of (16) and (17), (15) becomes

$$
\begin{aligned}
r_{12}^{-1} = {} & \tfrac{1}{2}\Sigma(1^\lambda)\cos\lambda V \\
& -\tfrac{1}{2}z r_1 r_2 \Sigma(3^\lambda)[\cos(\lambda+1)V-\cos(\lambda V+W)] \\
& +\tfrac{3}{8}z^2 r_1^2 r_2^2 \Sigma(5^\lambda)[2\cos\lambda V+\cos(\lambda+2)V+\cos(\lambda V+2W) \\
& \qquad -2\cos((\lambda+1)V-W)-2\cos((\lambda+1)V+W)] \\
& -\tfrac{5}{16}z^3 r_1^3 r_2^3 \Sigma(7^\lambda)[9\cos(\lambda+1)V-9\cos(\lambda V+W)\cos(\lambda+3)V \\
& \qquad -\cos(\lambda V+3W)-3\cos((\lambda+2)V+W) \\
& \qquad +3\cos((\lambda+1)V+2W)-3\cos((\lambda+2)V-W) \\
& \qquad +3\cos((\lambda+1)V-2W)].
\end{aligned}
$$

Exchanging wherever $(\lambda+1)$, $(\lambda+2)$, $(\lambda+3)$ appear the index λ by $(\lambda-1)$, $(\lambda-2)$, $(\lambda-3)$ and replacing λ by $-\lambda$, we can write in simple form

$$
\begin{aligned}
(19)\qquad r_{12}^{-1} = {} & \Sigma A^\lambda\cos\lambda V \\
& +\Sigma B^\lambda\cos(\lambda V+W) \\
& +\Sigma C^\lambda\cos(\lambda V+2W) \\
& +\Sigma D^\lambda\cos(\lambda V+3W) \\
& \quad \cdots\cdots\cdots\cdots
\end{aligned}
$$

where

$$
(20)\quad \begin{cases}
A^\lambda = \frac{1}{2}(1^\lambda) - \frac{z}{2} r_1 r_2 (3^{\lambda-1}) + \frac{3z^2}{8} r_1^2 r_2^2 [2(5^\lambda) + (5^{\lambda-2})] \\
\qquad\qquad - \frac{5z^3}{16} r_1^3 r_2^3 [9(7^{\lambda-1}) + (7^{\lambda-3})] + \dots \\
B^\lambda = \frac{z}{2} r_1 r_2 (3^\lambda) - \frac{3z^2}{4} r_1^2 r_2^2 [(5^{\lambda-1}) + (5^{\lambda+1})] \\
\qquad\qquad + \frac{5z^3}{16} r_1^3 r_2^3 [3(7^{\lambda+2}) + 9(7^\lambda) + 3(7^{\lambda-2})] + \dots \\
C^\lambda = \frac{3z^2}{8} r_1^2 r_2^2 (5^\lambda) - \frac{15z^3}{16} r_1^3 r_2^3 [(7^{\lambda+1}) + (7^{\lambda-1})] + \dots \\
D^\lambda = \frac{5z^3}{16} r_1^3 r_2^3 (7^\lambda) + \dots \\
\dots\dots\dots\dots
\end{cases}
$$

If the coefficients $A, B, \dots$ are solved from (20) and substituted in (19), we get finally, the following development

$$
(21)\qquad r_{12}^{-1} = \Sigma k_{a,\,\lambda,\,\beta} z^a \cos(\lambda V + \beta W)
$$

in which the k's are functions of r_1 and r_2 only, λ and β can take all positive and negative numbers and α is an entire positive number satisfying the condition

$$
a \geqq [\beta].
$$

The quantity β may, if we wish, be taken always positive or zero, since $\cos x = \cos(-x)$.

In the form (21) we select r_{12}^{-1} in order to consider it further. The k's are given functions of r_1 and r_2, therefore not constant. Let

$$
(22)\qquad r_1 = a_1(1+p_1), \quad r_2 = a_2(1+p_2)
$$

in which p_1 and p_2 are small quantities to be determined by (7), § 4. There are, therefore, double series of the form

$$
(22a)\qquad \begin{aligned} p_1 &= \Sigma k e_1^{\gamma_1} \cos(\delta_1 M_1), \\ p_2 &= \Sigma k e_2^{\gamma_2} \cos(\delta_2 M_2), \end{aligned}
$$

in which the k's are pure numbers, δ_1 and δ_2 all possible integral positive and negative numbers, and γ_1 and γ_2 positive numbers, for which according to § 4

$$
\gamma_1 - [\delta_1], \quad \gamma_2 - [\delta_2]
$$

are even and positive or 0.

Further, let

(23) $$v_1 = M_1 + q_1, \quad v_2 = M_2 + q_2,$$

so that q_1 and q_2 according to (11), §4, are likewise small quantities of the same character as p_1 and p_2. Then according to (8), (9) and (3)

(24) $$\begin{cases} V = V' + q_1 - q_2, \\ W = W' + q_1 + q_2, \end{cases}$$

in which

(25) $$\begin{cases} V' = M_1 + \pi_1 - \Omega_1 + \Pi_1 - (M_2 + \pi_2 - \Omega_2 + \Pi_2), \\ W' = M_1 + \pi_1 - \Omega_1 + \Pi_1 + (M_2 + \pi_2 - \Omega_2 + \Pi_2), \end{cases}$$

so that V' and W' are linear functions of the time.

Equation (21) now becomes

(26) $$r_{12}^{-1} = \Sigma k z^a \cos[\lambda V' + \beta W' + q_1(\lambda + \beta) + q_2(\beta - \lambda)].$$

If we put

(27) $$\begin{cases} \lambda + \beta = h_1, \\ \beta - \lambda = h_2, \\ M_1 + \pi_1 - \Omega_1 + \Pi_1 = l_1, \\ M_2 + \pi_2 - \Omega_2 + \Pi_2 = l_2, \end{cases}$$

the above equation becomes

(28) $$r_{12}^{-1} = \Sigma k z^a \cos(h_1 l_1 + h_2 l_2 + h_1 q_1 + h_2 q_2),$$

where, by (27), h_1 and h_2 are both even or both odd.

The condition $a \geqq [\beta]$ here becomes

(29) $$a \geqq [\tfrac{1}{2}(h_1 + h_2)].$$

Now, in order to further develop (28), we have to consider the four small quantities p_1, p_2, q_1, q_2, the first two appearing in the coefficients k, and the latter in the angles. They are best treated as follows: r_{12}^{-1} and and the k's are homogenous functions of r_1 and r_2 of the -1^{th} degree. If $k(a_1, a_2)$ is what k becomes when a_1 and a_2 are written in the places of r_1 and r_2, then

$$k(a_1, a_2) = \frac{1}{a_1} \cdot k\left(1, \frac{a_2}{a_1}\right),$$

and

$$\begin{aligned} k(r_1, r_2) &= k[a_1(1+p_1),\ a_2(1+p_2)] \\ &= \frac{1}{a_1(1+p_1)} \cdot k\left(1, \frac{a_2(1+p_2)}{a_1(1+p_1)}\right) \\ &= \frac{1}{1+p_1} \cdot k\left[a_1,\ a_2\left(1+\frac{p_2-p_1}{1+p_2}\right)\right]. \end{aligned}$$

If now, for brevity, k be written for $k(a_1, a_2)$, we have by Taylor's theorem

$$(30) \qquad \begin{aligned} k(r_1, r_2) = \frac{1}{1+p_1}\Bigg(& k + \frac{p_2-p_1}{1+p_1} \cdot a_2 \cdot \frac{\partial k}{\partial a_2} + \frac{\left(\frac{p_2-p_1}{1+p_1}\right)^2 a_2^2}{2!} \cdot \frac{\partial^2 k}{\partial a_2^2} \\ & + \frac{\left(\frac{p_2-p_1}{1+p_1}\right)^3 a_2^3}{3!} \frac{\partial^3 k}{\partial a_2^3} + \ldots \Bigg). \end{aligned}$$

If we wish to separate the elements of the two planets, we must develop the powers of (p_2-p_1) and obtain the two forms

$$\frac{p_1^r}{(1+p_1)^s} \text{ and } p_2^t,$$

in which r, s and t are positive integers and $r < s$. r and t form the degree with which the development of these forms in ascending powers of the eccentricities begins. We have also to consider the small quantities q_1 and q_2 which appear in (28). We have

$$(31) \begin{cases} \cos(h_1 l_1 + h_2 l_2 + h_1 q_1 + h_2 q_2) = \cos(h_1 l_1 + h_2 l_2)\cos(h_1 q_1 + h_2 q_2) \\ \qquad\qquad - \sin(h_1 l_1 + h_2 l_2)\sin(h_1 q_1 + h_2 q_2). \\ \text{and, further,} \\ \cos(h_1 q_1 + h_2 q_2) = \cos h_1 q_1 \cos h_2 q_2 - \sin h_1 q_1 \sin h_2 q_2, \\ \sin(h_1 q_1 + h_2 q_2) = \sin h_1 q_1 \cos h_2 q_2 + \cos h_1 q_1 \sin h_2 q_2. \end{cases}$$

If we substitute (30) and (31) in (28), we see that finally p_1, p_2, q_1, q_2 appear in the four following relations

$$(32) \qquad \begin{cases} \dfrac{p_1^r}{(1+p_1)^s} \cos h_1 q_1, & \dfrac{p_1^r}{(1+p_1)^s} \sin h_1 q_1, \\ p_2^t \cos h_2 q_2, & p_2^t \sin h_2 q_2. \end{cases}$$

These expressions, in which h_1 and h_2 are retained as undetermined integers, have been collected by Leverrier in tabular

form up to and including the seventh degree of the eccentricities. The developments of (32) follow the law given in (22a), the same as for the p's and q's. If we finally substitute them in any term of (28),

(33) $$kz^{a}\cos(h_1 l_1 + h_2 l_2 + h_1 q_1 + h_2 q_2),$$

and from this we get a four-fold infinite number of terms of the form

(34) $$\mathrm{K}z^{a} \ldots l_1^{\gamma_1} \,.\, l_2^{\gamma_2} \,.\, \cos(h_1 l_1 + h_2 l_2 + \delta_1 M_1 + \delta_2 M_2),$$

where a, h_1, h_2 remain unchanged as in (28), δ_1 and δ_2 take all possible positive and negative integral values, γ_1 and γ_2 are positive numbers satisfying the conditions that

$$\gamma_1 - [\delta_1] \text{ and } \gamma_2 - [\delta_1]$$

are even, positive, or zero.

In this way the perturbing function r_{12}^{-1} is to be developed in ascending powers of the quantities,

$$z = \sin^2 \tfrac{1}{2} J,\ e_1,\ e_2,$$

and in such a way that the angles

$$h_1 l_1 + h_2 l_2 + \delta_1 M_1 + \delta_2 M_2$$

increase proportionally to the time.

By the degree of (34) is meant the number

(35) $$g = 2a + \gamma_1 + \gamma_2.$$

Therefore, by (29) and (34)

(36) $$g - [h_1 + h_2 + \delta_1 + \delta_2]$$

must be even, positive or zero.

If we undertake to form systematically the term (33), a definite limit for the degree must be decided upon, beyond which the terms may be neglected.

The perturbing function

(37) $$r_{12}^{-1} = \Sigma K z^{a} e_1^{\gamma_1} e_2^{\gamma_2} \cos(h_1 l_1 + h_2 l_2 + \delta_1 M_1 + \delta_2 M_2)$$

may be expanded into a finite number of simple infinite series. For if n is the limit set for the degree g, then it follows from (35), that a, γ_1, γ_2 can contain only a finite number of systems

of values, and the same is true from (29) and (34) for h_1+h_2, δ_1, δ_2.

If we now put

$$h_1+h_2=2\Delta,\quad h_2=2\Delta-h_1,$$

and write simply h for h_2, then for a proper system of six numbers

(38) $$\alpha, \gamma_1, \gamma_2, \delta_1, \delta_2, \Delta,$$

the expression

(39) $$\sum_{h=-\infty}^{h=+\infty} Kz^{\alpha}.\, e_1^{\gamma_1}.\, e_2^{\gamma_2}.\, \cos\left[hl_1-(h-2\Delta)\,l_2+\delta_1 M_1+\delta_2 M_2\right],$$

where h passes through all integers from $-\infty$ to $+\infty$ while the system (38) remains unchanged gives such a simple infinite system as has been referred to; and, since, as we have seen, there are only a limited number of systems (38) for which the degree $g \overline{\overline{<}} n$, we have, if (39) is a term, broken the perturbing function into a limited number of terms. At the same time it must be noticed that the expression for k still contains the whole number h left as yet undetermined.

Moreover, it is not difficult to determine the number of terms in (39) whose degree is g. It is only necessary to determine the number Sg of the system (38) of the six whole numbers, where

(40) $$g=2\alpha+\gamma_1+\gamma_2$$

and where

(41) $$2\alpha-[2\Delta],\quad \gamma_1-[\delta_1],\quad \gamma_2-[\delta_2]$$

must be even, positive or 0, while Δ, δ_1, δ_2 can be positive or negative,

If α, γ_1, γ_2 are given, it follows from (41), that Δ can have besides $2\alpha+1$ values, α, $\alpha-1$, ... 1, 0, -1, ... $-\alpha$, and δ_1 also γ_1+1 values, $\gamma_1, \gamma_1-2, \ldots -\gamma_1$, and likewise δ_2, γ_2+1 values. The number Sq is, therefore,

$$Sq=\Sigma(2\alpha+1)(\gamma_1+1)(\gamma_2+1),$$

where a, γ_1, γ_2 run through all positive numbers including 0, which satisfy the relation (40).

If g is odd we can put

$$g-2a=2x+1=\gamma_1+\gamma_2,$$

where x may have all values from 0 to $\frac{1}{2}(g-1)$. Therefore

$$Sg=\sum_{x=0}^{x=\frac{1}{2}(g-1)}\ \sum_{\gamma_1=0}^{\gamma_1=2x+1}(\gamma_1+1)\,(2x+2-\gamma_1)\,(g-2x).$$

The series being taken with reference to γ_1, gives

$$Sg=2\sum_{x=1}^{x=\frac{1}{2}(g-1)}\frac{(x+1)\,(x+2)\,(2x+3)}{3}(g-2x).$$

The sum could of course be taken with reference to x, but this formula is most suited for the calculation of Sg when g is not too large.

If g is even, put

$$g-2a=2x=\gamma_1+\gamma_2,$$

so that x now has all values from 0 to $\frac{1}{2}g$. Then

$$Sg=\sum_{x=0}^{x=\frac{1}{2}g}\ \sum_{\gamma_1=0}^{\gamma_1=2x}(\gamma_1+1)\,(2x+1-\gamma_1)\,(g-2x+1),$$

or

$$Sg=\sum_{x=0}^{x=\frac{1}{2}g}\frac{(2x+1)\,(x+1)\,(2x+3)}{3}(g-2x+1).$$

The number of terms (39) of the degree g is, however, less than Sg. Since $\cos x=\cos(-x)$, each two systems (38) for which the three numbers Δ, δ_1, δ_2 have equal numerical values with opposite signs can be written as one, if h is replaced by $-h$ in the appropriate terms (39). The number sg of these terms is found to be half of the number of systems (38) for which, Δ, δ_1, δ_2 are not all 0 at the same time added to half the

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number for which they are zero. If we represent the number of the latter by Tg, then

$$s_g = \tfrac{1}{2}(Sg + Tg).$$

If g is odd, $Tg = 0$, for γ_1 and γ_2 can not be even at the same time as must be the case when Δ, δ_1, δ_2 are all zero. If g is even and we put $\Delta, \delta_1, \delta_2 = 0$, then $\gamma_1 = 2\gamma_1'$, $\gamma_2 = 2\gamma_2'$ and we have

$$\tfrac{1}{2}g = a + \gamma_1' + \gamma_2',$$

and Tg equals the number of the systems of solution of this Diophantine equation in positive, integral numbers including zero. That is

$$T_g = \tfrac{1}{2}(\tfrac{1}{2}g + 1)(\tfrac{1}{2}g + 2).$$

We see, therefore, that

(1). If g is odd

$$s_g = \sum_{x=0}^{x=\frac{1}{2}(g-1)} \frac{(x+1)(x+2)(2x+3)}{3}(g-2x),$$

(2). If g is even

$$s_g = \tfrac{1}{2}\{\tfrac{1}{2}(\tfrac{1}{2}g+1)(\tfrac{1}{2}g+2)\}$$

$$+\sum_{x=0}^{x=\frac{1}{2}g} \frac{(2x+1)(x+1)(2x+3)}{3}(g-2x+1).$$

The number of terms (39) for the different degrees, as obtained from the above, is

Degree	0,	1,	2,	3,	4,	5,	6,	7,	...
Terms	1,	2,	8,	16,	38,	68,	128,	208,	...

Up to the seventh degree inclusive, there are 469 terms to compute and this was done by Leverrier. If we realize all that is to be calculated before we reach the coefficients k as functions a_1, a_2 and of the undetermined whole number h, we shall get a fair idea of the great amount of labor involved, though this labor can be reduced about a third. The magnitude of the calculations can be judged by the fact that the results above fill

53 quarto pages. Since I have sufficiently indicated the general character of these computations, I will refrain from further details by referring the reader to the work itself. I have wished only to give him a bird's-eye view of the vast amount of calculation needed to bring the planetary tables up to their present stage of perfection.

The other part of the perturbing function, viz:

$$\frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{r_2^3} = \frac{r_1}{r_2^2}(\cos V \cos^2 \tfrac{1}{2} J + \cos W \sin^2 \tfrac{1}{2} J),$$

can be developed in the same manner as r_{12}^{-1}. Indeed the development is notably simpler than the preceding because the only values the numbers α can have are 0 or 2, and the only values h_1 and h_2 can have are ± 1, while otherwise the form of the terms remains the same.

In a memoir entitled *Mémoire sur le développement en séries des coordonnées des planètes et de la fonction perturbatrice* (*Journal de Liouville*, 1860), Puiseux has given formulas for the computation of the coefficients, K for every value of the numbers $\alpha, \gamma_1, \gamma_2, h_1, h_2, \delta_1, \delta_2$. The operations by which his formulas were obtained have been for the most part, only indicated; by him they were carried out to the minutest detail. They represent the entire formal solution of the problem when his five quantities $c, \epsilon, \epsilon', \omega, \omega'$ have been developed in ascending powers of z, e_1, e_2. They become, however, more complicated than they otherwise are, and I doubt if it is practicable to carry out the computations for the higher degrees, as Leverrier has succeeded in doing, up to and including the seventh. The formulas are perhaps not the simplest, for possibly many of the summations may admit of contraction. In any case, Puiseux's expressions, for the very reason that they are expeditious, do not readily permit the deduction of the relations of which we are to speak at the end of this section.

However convenient for practical computation may be the form of development given here, yet not all of the twelve elements,

$$a_1, a_2, e_1, e_2, i_1, i_2, \pi_1, \pi_2, \Omega_1, \Omega_2, \zeta_1, \zeta_2,$$

appear in it in a recognizable form. (For simplicity of expression the mean longitudes ζ_1, ζ_2 are included among the elements, although they do not contain the time). In fact the last eight of the elements are implicity contained in J, l_1, l_2, M_1, M_2 and in the following manner:

$$\cos J = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos(\Omega_1 - \Omega_2),$$
$$l_1 = \zeta_1 - \Omega_1 + \Pi_1, \quad l_2 = \zeta_2 - \Omega_2 + \Pi_2,$$
$$M_1 = \zeta_1 - \pi_1, \qquad M_2 = \zeta_2 - \pi_2,$$

where Π_1 and Π_2 are to be determined from (6). If the inclinations, longitudes of perihelia, longitudes of the nodes, and the mean longitudes are to be expressed explicitly in the perturbing function, each term of the perturbing function

(42) $$Kz^{\alpha} e_1^{\gamma_1} e_2^{\gamma_2} \cos(h_1 l_1 + h_2 l_2 + \delta_1 M_1 + \delta_2 M_2),$$

or

(43) $$Kz^{\alpha} e_1^{\gamma_1} e_2^{\gamma_2} \cos(A + px + qy),$$

where for brevity,

(44) $$\begin{cases} A = (h_1 + \delta_1)\zeta_1 + (h_2 + \delta_2)\zeta_2 - \delta_1 \pi_1 - \delta_2 \pi_2, \\ x = -\Pi_1 + \Pi_2 + \Omega_1 - \Omega_2, \quad p = -\frac{1}{2}(h_1 + h_2), \\ y = -\Pi_1 - \Pi_2 + \Omega_1 + \Omega_2, \quad q = -\frac{1}{2}(h_1 - h_2), \end{cases}$$

must be expanded in a three-fold infinite series.

The three angles J, Π_1, Π_2 are to be eliminated by the substitution of their values from (5) and (6) and the result reduced to a suitable form, which can be done in the following way.

From (6) it follows that

(45) $$\begin{cases} e^{\pm ix} = (a + b e^{\pm i(\Omega_1 - \Omega_2)})^2, \\ e^{\pm iy} = (a_1 e^{\pm i\Omega_1} + b_1 e^{\pm i\Omega_2})^2, \text{ in which} \\ a = \dfrac{\cos\frac{1}{2}i_1 \cos\frac{1}{2}i_2}{\cos\frac{1}{2}J}, \qquad b = \dfrac{\sin\frac{1}{2}i_1 \sin\frac{1}{2}i_2}{\cos\frac{1}{2}J}, \\ a_1 = \dfrac{\sin\frac{1}{2}i_1 \cos\frac{1}{2}i_2}{\sin\frac{1}{2}J}, \qquad b_1 = -\dfrac{\cos\frac{1}{2}i_1 \sin\frac{1}{2}i_2}{\sin\frac{1}{2}J}, \end{cases}$$

in which $i = \sqrt{-1}$ and $e = 2.7182818284$,

and, therefore,

$$e^{\pm pix} = \sum \frac{[2p]!}{[\lambda]!\,[\mu]!} a^{[\lambda]} b^{[\mu]} e^{\pm\mu(\Omega_1 - \Omega_2)},$$

where λ and μ run through all whole numbers which have the same sign as p (0 included), for which

$$\lambda + \mu = 2p.$$

Likewise

$$e^{\pm qiy} = \sum \frac{[2q]!}{[\lambda_1]!\,[\mu_1]!} a_1^{[\lambda_1]} b_1^{[\mu_1]} e^{\pm(\lambda_1 \Omega_1 + \mu_1 \Omega_2)},$$

where λ_1 and μ_1 run through all whole numbers which have the same sign as q (0 included), for which

$$\lambda_1 + \mu_1 = 2q = -h_1 - h_2.$$

From this it follows, finally, that

(46) $$K z^a e_1^{\gamma_1} e_2^{\gamma_2} \cos(A + px + qy)$$

$$= \sum K \frac{[2p]!\,[2q]!}{[\lambda]!\,[\mu]!\,[\lambda_1]!\,[\mu_1]!} e_1^{\gamma_1} e_2^{\gamma_2} z^a a^{[\lambda]} b^{[\mu]} a^{[\lambda_1]} b^{[\mu_1]} \times$$
$$\cos[A + \mu(\Omega_1 - \Omega_2) + \lambda_1 \Omega_1 + \mu_1 \Omega_2].$$

If we now put

$$\mathfrak{a}_1 = h_1 + \delta_1, \quad \mathfrak{b}_1 = -\delta_1, \quad \mathfrak{c}_1 = \mu + \lambda_1,$$
$$\mathfrak{a}_2 = h_2 + \delta_2, \quad \mathfrak{b}_2 = -\delta_2, \quad \mathfrak{c}_2 = -\mu + \mu_1,$$

so that

(47) $$\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{b}_1 + \mathfrak{b}_2 + \mathfrak{c}_1 + \mathfrak{c}_2 = 0,$$

the angle in the second member of (46) becomes

(48) $$\mathfrak{a}_1 \zeta_1 + \mathfrak{a}_2 \zeta_2 + \mathfrak{b}_1 \pi_1 + \mathfrak{b}_2 \pi_2 + \mathfrak{c}_1 \Omega_1 + \mathfrak{c}_2 \Omega_2 = L.$$

By the help of equation (45) the product $z^a a^{[\lambda]} b^{[\mu]} a_1^{[\lambda_1]} b_1^{[\mu_1]}$ becomes

$$(-1)^{\mu_1} \frac{z^{a-[q]}}{(1-z)^{[p]}} (\cos \tfrac{1}{2} i_1)^{[\lambda] + [\mu_1]} (\cos \tfrac{1}{2} i_2)^{[\lambda] + [\lambda_1]} \times$$
$$(\sin \tfrac{1}{2} i_1)^{[\mu] + [\lambda_1]} (\sin \tfrac{1}{2} i_2)^{[\mu] + [\mu_1]}.$$

The product of the last four factors can be developed in ascending powers of i_1 and i_2. The first factor $\frac{z^{a-[q]}}{(1-z)^{[p]}}$ can, since $a - [q]$ is not negative, be developed in positive and ascending powers of z. But z itself and its powers can be at once arranged in cosines of multiples of $\Omega_1 - \Omega_2$ and the coefficients become functions of i_1 and i_2 and can be easily arranged

in ascending powers of these quantities. If these are all introduced into the above product and this again in the term (46), the development of the perturbing function finally receives the following form:

$$r_{12}^{-1} = \Sigma\, K e_1^{\gamma_1} e_2^{\gamma_2} i_1^{\alpha_1} i_2^{\alpha_2} \cos L. \tag{49}$$

L has here the the form (48) in which the whole numbers $\mathfrak{a}_1, \ldots$ satisfy (47) but otherwise can take all positive and negative values, except that $\mathfrak{c}_1 + \mathfrak{c}_2$ must always be even. The exponents $\gamma_1, \gamma_2, \alpha_1, \alpha_2$ are all positive and satisfy the conditions that

$$\gamma_1 - [\mathfrak{b}_1], \quad \gamma_2 = [\mathfrak{b}_2], \quad \alpha_1 - [\mathfrak{c}_1], \quad \alpha_2 - [\mathfrak{c}_2]$$

must all be even, positive or zero. The coefficients K are homogenous functions of the order -1 of a_1 and a_2, in whose general expressions also enter the above whole numbers. The degree g of a term is here

$$g = \gamma_1 + \gamma_2 + \alpha_1 + \alpha_2 \geqq [\mathfrak{a}_1 + \mathfrak{a}_2].$$

The second part (2) of the perturbing function gives terms of the same form as (49), but in this case $[\mathfrak{c}_1] + [\mathfrak{c}_2] = 0$ or 2. The computation of this part is relatively simple.

If we put $\mathfrak{a}_1 + \mathfrak{a}_2 = \Delta$, there is only a finite number of the systems of numbers

$$\gamma_1, \gamma_2, \alpha_1, \alpha_2, \mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{c}_1, \mathfrak{c}_2, \Delta,$$

for which g does not surpass a certain limit.

If, with restriction to such a limit, the perturbing function is resolved into simple series, of which each consists of an infinite number of terms agreeing with the above system of numbers, the number of series is limited. Their number s_g for a given degree can be easily obtained by the formulas

$$s_{2k} = u_{2k} + \frac{4.3.2}{1.2.3} u_{2k-2} + \frac{5.4.3}{1.2.3} u_{2k-4} + \ldots + \frac{(4+k-1)(4+k-2)(4+k-3)}{1.2.3} u_0,$$

$$s_{2k+1} = u_{2k+1} + \frac{4.3.2}{1.2.3} u_{2k-1} + \frac{5.4.3}{1.2.3} u_{2k-3} + \ldots + \frac{(4+k-1)(4+k-2)(4+k-3)}{1.2.3} u_1,$$

where, in general,

$$u_k = \frac{2k(k^2+2)}{3},$$

except for $k=0$, for which

$$u_0 = 1.$$

From these we find

$$\begin{array}{llll} s_0 = 1, & s_2 = 12, & s_4 = 90, & s_6 = 444, \\ s_1 = 2, & s_3 = 30, & s_5 = 222, & s_7 = 858. \end{array}$$

If Leverrier had selected the complete expeditious form (49), he would have had no less than 1659 terms to compute.

Many mathematicians have occupied themselves with the development of the perturbing function. Cauchy introduced the eccentric anomalies E_1, E_2 and developed in terms of the cosines of their multiples. Bessel's functions then enabled him easily to substitute the mean for the eccentric anomalies. His numerous investigations on this subject appeared in the *Comptes rendus* and are collected in his *Oeuvres complètes.*

While the methods hitherto used by astronomers for the development of

$$r_{12}^{-1} = \frac{1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}$$

are direct, it does not follow that this development may not be reached indirectly. For brevity, put $F = r_{12}^{-1}$. An indirect method of development would be that of forming certain differential equations which are satisfied by F as a function of the twelve elements of the two planets and which serve to determine F.

If we consider F as a function of the coordinates $x_1, x_2, \ldots$ and of the component velocities $u_1, u_2, \ldots$, then F will satisfy the six partial differential equations

$$(50) \qquad \begin{cases} \dfrac{\partial F}{\partial u_1} = 0, & \dfrac{\partial F}{\partial v_1} = 0, \quad \dfrac{\partial F}{\partial w_1} = 0, \\ \dfrac{\partial F}{\partial u_2} = 0, & \dfrac{\partial F}{\partial v_2} = 0, \quad \dfrac{\partial F}{\partial w_2} = 0, \end{cases}$$

which express the fact that F is independent of the component

velocities. We have also the following six partial differential equations

$$(51)\quad \begin{cases} \dfrac{\partial F}{\partial x_1}+\dfrac{\partial F}{\partial x_2}=0, \quad \dfrac{\partial F}{\partial y_1}+\dfrac{\partial F}{\partial y_2}=0, \quad \dfrac{\partial F}{\partial z_1}+\dfrac{\partial F}{\partial z_2}=0, \\ x_1\dfrac{\partial F}{\partial y_1}-y_1\dfrac{\partial F}{\partial x_1}+x_2\dfrac{\partial F}{\partial y_2}-y_2\dfrac{\partial F}{\partial x_2}=0, \\ y_1\dfrac{\partial F}{\partial z_1}-z_1\dfrac{\partial F}{\partial y_1}+y_2\dfrac{\partial F}{\partial z_2}-z_2\dfrac{\partial F}{\partial y_2}=0, \\ z_1\dfrac{\partial F}{\partial x_1}-x_1\dfrac{\partial F}{\partial z_1}+z_2\dfrac{\partial F}{\partial x_2}-x_2\dfrac{\partial F}{\partial z_2}=0, \end{cases}$$

of which the first three show that F depends on x_1-x_2, y_1-y_2, z_1-z_2 alone, and the last three that F is a function of r_{12}. The six equations (51) are not independent, an identical relation exists between them. Finally, as a homogenous function of the coordinates of the order -1, F satisfies the partial differential equation

$$(52)\quad x_1\frac{\partial F}{\partial x_1}+y_1\frac{\partial F}{\partial y_1}+z_1\frac{\partial F}{\partial z_1}+x_2\frac{\partial F}{\partial x_2}+y_2\frac{\partial F}{\partial y_2}+z_2\frac{\partial F}{\partial z_2}=-F.$$

Equations (50), (51), (52) determine F with the exception of a constant factor c, and from them it follows that

$$F=cr_{12}^{-1}.$$

If, now, the elements are substituted for the component velocities, the partial differential equations pass into others having the element $a_1, \ldots \zeta_1, a_2, \ldots \zeta_2$ as the unknown quantities. For example, the equation

$$\frac{\partial F}{\partial u_1}=0$$

takes the form

$$\frac{\partial F}{\partial a_1}\frac{\partial a_1}{\partial u_1}+\frac{\partial F}{\partial e_1}\frac{\partial e_1}{\partial u_1}+\frac{\partial F}{\partial i_1}\frac{\partial i_1}{\partial u_1}+\frac{\partial F}{\partial \pi_1}\frac{\partial \pi_1}{\partial u_1}+\frac{\partial F}{\partial \Omega_1}\frac{\partial \Omega_1}{\partial u_1}+\frac{\partial F}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial u_1}=0,$$

in which the coefficients $\dfrac{\partial a_1}{\partial u_1}, \dfrac{\partial e_1}{\partial u_1} \ldots$, are to be considered functions of $a_1, e_1, \ldots$. If for F we now take the form (49), the thus modified partial differential equations will determine K up to a single constant factor which remains over but which can then be easily determined.

Two of the partial differential equations can be at once written out. One is (52) which now takes the form

$$a_1\frac{\partial F}{\partial a_1}+a_2\frac{\partial F}{\partial a_2}=-F,$$

and from which it follows that every coefficient K satisfies this differential equation. The other is a combination of the two last of equations (51) and takes the form

$$\frac{\partial F}{\partial \zeta_1}+\frac{\partial F}{\partial \zeta_2}+\frac{\partial F}{\partial \pi_1}+\frac{\partial F}{\partial \pi_2}+\frac{\partial F}{\partial \Omega_1}+\frac{\partial F}{\partial \Omega_2}=0,$$

from which the equation of condition (47) follows for each term of (48). The remaining ten equations (50) and (51) are somewhat more complicated for we know that the K homogeneous functions of the degree -1 in a_1 and a_2 serve to determine these functions and to form an infinite number of relations between them and their partial derivatives with respect to a_1 and a_2.

The very expeditions form (49) is often unnecessary. When it is only necessary to represent the perturbing function as an analytical function of t which is contained in ζ_1 and ζ_2 only, we can collect all the terms of (49) for which $\mathfrak{a}_1$ and $\mathfrak{a}_2$ have the same values, and the form becomes

$$(53)\quad r_{12}^{-1}=\tfrac{1}{2}\Sigma\{A_{\mathfrak{a}_1,\,\mathfrak{a}_2}\cos(\mathfrak{a}_1\zeta_1+\mathfrak{a}_2\zeta_2)+B_{\mathfrak{a}_1,\,\mathfrak{a}_2}\sin(\mathfrak{a}_1\zeta_1+\mathfrak{a}_2\zeta_2)\},$$

in which $\mathfrak{a}_1$ and $\mathfrak{a}_2$ denote all positive and negative numbers and it is assumed that

$$A_{-\mathfrak{a}_1,\,-\mathfrak{a}_2}=A_{\mathfrak{a}_1,\,\mathfrak{a}_2},\quad B_{-\mathfrak{a}_1,\,-\mathfrak{a}_2}=B_{\,1,\,\mathfrak{a}_2}.$$

The coefficients A and B can be at once expressed in double definite integrals, if we employ the process used by Fourier for the trigonometrical functions of an angle. We have

$$A_{\mathfrak{a}_1,\,\mathfrak{a}_2}=\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}r_{12}^{-1}\cos(\mathfrak{a}_1\zeta_1+\mathfrak{a}_2\zeta_2)\,d\zeta_1\,d\zeta_2,$$

$$B_{\mathfrak{a}_1,\,\mathfrak{a}_2}=\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}r_{12}^{-1}\sin(\mathfrak{a}_1\zeta_1+\mathfrak{a}_2\zeta_2)\,d\zeta_1\,d\zeta_2.$$

The numerical computation of these coefficients can be performed in two ways. On the one hand, after Liouville, (*Sur le calcul des, inégalités périodiques, Jour. de Math. I, 1836*), we may expand the double integrals into a very rapidly converging series of simple integrals which we may determine by mechanical quadrature. Or, we may, after Leverrier (*Recherches astronomique*) employ a process of interpolation. Finally, in the articles already referred to in *Comptes rendus*, for the case in which $[\mathfrak{a}_1+\mathfrak{a}_2]$ is a large number, Cauchy has given a special process which he has applied practically to a definite term in the theory of Pallas and Jupiter.

Finally we will also notice that Hansen has given an entirely novel manner of developing the perturbing function. It consists in using the mean anomaly of one body and the true anomaly of the other when the latter, as is the case with the comets and some of the planets, etc., has an eccentricity so large that the series is slowly convergent or entirely fails. This method of development has its chief value in relation to the moon, the asteriods and comets. It will not be given here, since we are considering only the sun and major planets.

25. THE DEVELOPMENT OF $(a_1^2-2a_1a_2\cos\delta+a_2^2)^{-\frac{1}{2}s}$ IN A TRIGONOMETRIC SERIES.

In the method used in the preceding section for the analytical development of the reciprocal of the distance between two planets, the coefficients (s^i) of the function

$$(1)\quad (a_1^2-2a_1a_2\cos\delta+a_2^2)^{-\frac{1}{2}s}=\tfrac{1}{2}(s^0)+(s^1)\cos\delta+(s^2)\cos 2\delta+\ldots$$
$$=\tfrac{1}{2}\sum_{i=-\infty}^{i=+\infty}(s^i)\cos i\delta,$$

play a prominent part. To justify the second manner of writing the second member of (1) and to define (s^i) also for negative values of i, we suppose that

$$(2)\qquad (s^i)=(s^{-i}).$$

By the aid of Fourier's theorem for the development in trigonometric series, the coefficients (s^i) can be brought into the form

$$(3) \qquad (s^i) = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos i\,\delta\, d\delta}{(\sqrt{a_1^2 - 2a_1 a_2 \cos\delta + a_2^2})^s},$$

which, by putting $\cos\delta = x$, can be treated as an elliptic integral. Laplace proposed a direct method which is better. It is

$$a_1^2 + a_2^2 - 2a_1 a_2 \cos\delta = (a_1 - a_2 e^{i\delta})\,(a_1 - a_2 e^{-i\delta}),$$

or if we put

$$\frac{a_2}{a_1} = \alpha,$$

$$(4) \quad (a_1^2 + a_2^2 - 2a_1 a_2 \cos\delta)^{-\frac{1}{2}s} = a_1^{-s}(1 - \alpha e^{i\delta})^{-\frac{1}{2}s}(1 - \alpha e^{-i\delta})^{-\frac{1}{2}s}.$$

If $\alpha < 1$, (in case it is not, we can make it so by exchanging a_1 and a_2), the binomial theorem gives

$$\begin{aligned}(a_1^2 - 2a_1 a_2 \cos\delta + a_2^2)^{-\frac{1}{2}s} \\ a_1^{-s}\Big(1 + \frac{s}{2}\alpha e^{i\delta} + \frac{s(s+2)}{2^2\,.\,2!}\alpha^2 e^{2i\delta} \\ + \frac{s(s+2)(s+4)}{2^3\,.\,3!}\alpha^3 e^{3i\delta} + \dots\Big)\times \\ \Big(1 + \frac{s}{2}\alpha e^{-i\delta} + \frac{s(s+2)}{2^2\,.\,2!}\alpha^2 e^{-2i\delta} \\ + \frac{s(s+2)(s+4)}{2^3\,.\,3!}\alpha^3 e^{-3i\delta} + \dots\Big).\end{aligned}$$

If the two series are multiplied together and the product arranged in ascending powers of α, the coefficients will be entire functions of $x = \cos\delta$. These functions play an important part in many problems and are called spherical harmonics. But if they are arranged in powers of $e^{i\delta}$, they give the coefficients (s^i) to be used here. They are

$$\begin{aligned}(5) \qquad \tfrac{1}{2}(s^i) = a_1^{-s}\Big(&\frac{s(s+2)\dots(s+2i-2)}{2^i i!}\alpha^i \\ &+ \frac{s(s+2)\dots(s+2i)}{2^{i+1}(i+1)!}\,\frac{s}{2}\,\alpha^{i+2} \\ &+ \frac{s(s+2)\dots(s+2i+2)}{2^{i+2}(i+2)!}\,\frac{s(s+2)}{2^2\,.\,2!}\,\alpha^{i+4} \\ &+ \dots\dots\dots\dots\Big).\end{aligned}$$

This is the general expression (s^i) for every value of s and i. (For negative values of i it is illusory, but in that case the equation (2) is to be used). But it is not yet sufficient, we must yet obtain a series of fundamental relations between the quantities (s^i). This is given very simply by a reduction of (s^i) by Gauss's hypergeometric series,

$$(6)\quad F(\alpha,\beta,\gamma,x)=1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\dots$$

This is

$$(7)\qquad \tfrac{1}{2}(s^i)=a_1^{-s}.\frac{s(s+2)(s+4)\dots(s+2i-2)}{2^i.i!}a^i\times F(\tfrac{1}{2}s,\tfrac{1}{2}s+i,i+1,a^2).$$

However, the required relations are given with the same ease by equation (1). If, for brevity, we put

$$a_1^2-2a_1a_2\cos\delta+a_2^2=\rho,$$

we obtain, by differentiating (1) with respect to δ,

$$-sa_1a_2\rho^{-\frac{1}{2}(s+2)}\sin\delta=-\tfrac{1}{2}\sum_{-\infty}^{+\infty}i(s^i)\sin i\delta,$$

or, if both members are multiplied by 2ρ

$$2s\,a_1a_2\sin\delta\tfrac{1}{2}\Sigma(s^i)\cos i\delta$$

$$=(a_1^2+a_2^2)\sum_{-\infty}^{+\infty}i(s^i)\sin i\delta-2a_1a_2\cos\delta\sum_{-\infty}^{+\infty}i(s^i)\sin i\delta.$$

But

$$\sin\delta\,\Sigma(s^i)\cos i\delta=\tfrac{1}{2}\Sigma(s^i)\sin(i+1)\delta-\tfrac{1}{2}\Sigma(s^i)\sin(i-1)\delta,$$

and if in the second member the summation index i be exchanged with $(i-1)$ and correspondingly with $(i+1)$ it follows that

$$\sin\delta\,\Sigma(s^i)\cos i\delta=\tfrac{1}{2}\Sigma[(s^{i-1})-(s^{i+1})]\sin i\delta.$$

In the same manner

$$\cos\delta\,\Sigma i(s^i)\sin i\delta=\tfrac{1}{2}\Sigma[(i-1)(s^{i-1})+(i+1)(s^{i+1})]\sin i\delta,$$

the summations always extending from $-\infty$ to $+\infty$. Therefore

$$sa_1 a_2 \,.\, \tfrac{1}{2}\Sigma[(s^{i-1})-(s^{i+1})]\sin i\delta$$
$$=\Sigma[(a_1^2+a_2^2)i(s^i)-a_1 a_2((i-1)(s^{i-1})+(i+1)(s^{i+1}))]\sin i\delta.$$

The coefficient of $\sin i\delta$ in the first member is, according to (2), changed by replacing i by $-i$ and the same is true in the second member. The coefficients must therefore be equal. Hence

$$\tfrac{1}{2}sa_1 a_2[(s^{i-1})-(s^{i+1})]$$
$$=(a_1^2+a_2^2)i(s^i)-a_1 a_2[(i-1)(s^{i-1})+(i+1)(s^{i+1})]. \tag{8}$$

If we put

$$\frac{a_1^2+a_2^2}{a_1 a_2}=a+\frac{1}{a}=k, \tag{9}$$

it follows that

$$(s^{i+1})=\frac{2ki(s^i)-(2i+s-2)(s^{i-1})}{-s+2i+2}. \tag{10}$$

By means of this formula any (s^i) may be obtained from the two immediately preceding. For example, from (s^0) and (s^1) we have

$$(s^2)=\frac{2k(s^1)-s(s^0)}{-s+4},$$
$$(s^3)=\frac{4k(s^2)-(s+2)(s^1)}{-s+6}$$
$$=\frac{[8k^2-(s+2)(s-4)](s^1)-4sk(s^0)}{(-s+4)(-s+6)}.$$

The equation (10) affords the relations between the coefficients of a single series (1). There are also such relations between the coefficients of two such series for which the s's differ by an even number. These are sufficient, for in astronomy, the only cases which occur are those in which s is odd.

Multiplying (1) by ρ, we get

$$\rho^{-\frac{1}{2}(s-2)}=\tfrac{1}{2}\Sigma((s-2)^i)\cos i\delta=(a_1^2+a_2^2)\tfrac{1}{2}\Sigma(s^i)\cos i\delta$$
$$-a_1 a_2[\tfrac{1}{2}\Sigma(s^i)\cos(i+1)\delta+\tfrac{1}{2}\Sigma(s^i)\cos(i-1)\delta]$$
$$=\tfrac{1}{2}\Sigma\{(a_1^2+a_2^2)(s^i)-a_1 a_2(s^{i-1})+(s^{i+1})\}\cos i\delta.$$

Since, by (2), the coefficient in braces remains unchanged when i is exchanged with $-i$, we must have

$$(12) \qquad ((s-2)^i) = a_1 a_2 [k(s^i) - (s^{i-1}) - (s^{i+1})],$$

or, by (10),

$$(13) \qquad ((s-2)^i) = (-s+2) a_1 a_2 \frac{k(s^i) - 2(s^{i-1})}{-s+2i+2}.$$

In order, conversely, to express the quantities $(s)^i$ by $((s-2)^i)$, exchange i with $-(i-1)$ in (12) and use (2). Then

$$(14) \qquad ((s-2)^{i-1}) = (-s+2) a_1 a_2 \frac{k(s^{i-1}) - 2(s^i)}{-s-2i+4}.$$

It follows from (13) and (14), that

$$(15) \qquad (s^i) - (s^{i-1}) = \frac{(-s+2i+2)((s-2)^i)}{(-s+2) a_1 a_2 (k+2)} + \frac{(s+2i-4)((s-2)^{i-1})}{(-s+2) a_1 a_2 (k+2)}.$$

From this an important conclusion may be drawn. If s is positive, (s^i) is, by (5), positive for every value of i. If $s > 2$, then $((s-2)^i)$ and $((s-2)^{i-1})$ are positive. Therefore, if we select i positive and so great that both $(-s+2i+2)$ and $(s+2i-4)$ are positive the second member of (15) becomes negative and

$$(s^i) < (s^{i-1}).$$

If we put, for example, $s = 3$, $i = 2$, it follows

$$(3^2) - (3^1) = -\frac{3[(1^2) + (1^1)]}{a_1 a_2 (k+2)},$$

hence

$$(16) \qquad (3^2) < (3^1).$$

This inequality will have an important application in the following paragraphs.

If we solve (13) and (14) for (s^i) and (s^{i-1}), we get

$$(17) \begin{cases} (s^i) = \dfrac{k(-s+2i+2)((s-2)^i) + 2(-s-2i+4)((s-2)^{i-1})}{(-s+2) a_1 a_2 (k^2-4)}, \\[2ex] (s^{i-1}) = \dfrac{2(-s+2i+2)((s-2)^i) + k(-s-2i+4)((s-2)^{i-1})}{(-s+2) a_1 a_2 (k^2-4)}. \end{cases}$$

Finally we will show how to express the derivatives of (s^i) with respect to a_1 and a_2 in terms of (s^i). If we differentiate (1) with respect to a_1, we get

$$-s\rho^{-\frac{1}{2}(s+2)}(a_1-a_2\cos\delta)=\tfrac{1}{2}\sum\frac{\partial(s^i)}{\partial a_1}\cos i\delta,$$

or

$$\tfrac{1}{2}\sum\left\{-s\left(a_1((s+2)^i)-a_2\frac{(s+2)^{i-1}+((s+2)^{i+1})}{2}\right)\right\}\cos i\delta$$

$$=\tfrac{1}{2}\sum\frac{\partial(s^i)}{\partial a_1}\cos i\delta,$$

and, therefore,

$$\frac{\partial(s^i)}{\partial a_1}=-sa_1((s+2)^i)+sa_2\frac{((s+2)^{i-1})+((s+2)^{i+1})}{2},$$

and also by (12), if we write $(s+2)$ for s

$$\frac{\partial(s^i)}{\partial a_1}=-\frac{s}{2a_1}\{(a_1^2-a_2^2)((s+2)^i)+(s^i)\}, \tag{18}$$

and, finally, by (17),

$$\frac{\partial(s^i)}{\partial a_1}=\frac{[-sa_1^2+i(a_1^2+a_2^2)](s^i)+a_1a_2(-s-2i+2)(s^{i-1})}{a_1(a_1^2-a_2^2)}, \tag{19}$$

By exchanging the subscripts 1 and 2, we get the derivatives $\frac{\partial(s^i)}{\partial a_2}$. The derivatives of a_2 can also be obtained from those with respect to a_1 from the fact that (s^i) is a homogeneous function of the $(-s)^{\text{th}}$ degree in a_1 and a_2, and hence the partial differential equation

$$a_1\frac{\partial(s^i)}{\partial a_1}+a_2\frac{\partial(s^i)}{\partial a_2}=-s(s^i) \tag{20}$$

must be satisfied.

26. The Terms of the Perturbing Function of the Degrees 0, 1 and 2.

The analytical development of the coefficents of the perturbing function requires, if we pass beyond the second degree, days, weeks and even months of labor in computing them. If the computation is confined to the zero, first and second degrees, the necessary computations are not excessive and the reader can, for himself, easily verify the formulas.

The equation (15), § 24, then becomes

$$(1) \qquad r_{12}^{-1} = \rho^{-\frac{1}{2}} - r_1 r_2 (\cos V - \cos W) z \rho^{-\frac{3}{2}}$$
$$= \tfrac{1}{2}\Sigma(1^\lambda)\cos\lambda V - \tfrac{1}{2} r_1 r_2 z \Sigma(3^\lambda)\cos(\lambda+1)V$$
$$+ \tfrac{1}{2} r_1 r_2 z \Sigma(3^\lambda)\cos(\lambda V + W).$$

The quantities p and q become

$$p = -e\cos M + \tfrac{1}{2}e^2(1-\cos 2M),$$
$$q = 2e\sin M + \tfrac{5}{4}e^2 \sin 2M.$$

The (s^λ) are here functions of r_1 and r_2 and may be denoted by $(s^\lambda)_{r_1,\,r_2}$. On the other hand (s^λ) is the $(s^\lambda)_{a_1,\,a_2}$. The equation (1), in which p and q need be substituted only in the first term of the second member, now becomes

$$r_{12}^{-1} = \tfrac{1}{2}\Sigma(1^\lambda)_{a_1(1+p),\,a_2(1+p_2)}\cos[\lambda(l_1-l_2)+\lambda(q_1-q_2)]$$
$$-\tfrac{1}{2}a_1 a_2 z \Sigma(3^\lambda)\cos[(\lambda+1)(l_1-l_2)]$$
$$+\tfrac{1}{2}a_1 a_2 z \Sigma(3^\lambda)\cos[(\lambda+1)l_1-(\lambda-1)l_2].$$

If this is now developed in ascending powers of p_1, p_2, q_1, q_2 and then their values given above are substituted, it follows that

$$r_{12}^{-1} = \sum_{\lambda=-\infty}^{\lambda=+\infty}[A_\lambda\cos\varphi + e_1 B_\lambda\cos(\varphi+M_1) + e_2 C_\lambda\cos(\varphi+M_1)$$
$$+ e_1^2 D_\lambda\cos\varphi + e_2^2 E_\lambda\cos\varphi + e_1 e_2 F_\lambda\cos(\varphi+M_1-M_2)$$
$$+ z\, G_\lambda\cos\varphi + e_1^2 H_\lambda\cos(\varphi+2M_1) + e_2^2 J_\lambda\cos(\varphi+2M_2)$$
$$+ e_1 e_2 K_\lambda\cos(\varphi+M_1+M_2) + z L_\lambda\cos(\varphi+l_1+l_2)],$$

where for brevity,

$$(2) \qquad \varphi = \lambda(l_1-l_2),$$

and where the coefficients have the following values

$$(3) \qquad \begin{cases} A_\lambda = \tfrac{1}{2}I_\lambda, \\ B_\lambda = \tfrac{1}{2}(-I_\lambda' + 2\lambda I_\lambda), \\ C_\lambda = \tfrac{1}{2}(-'I_\lambda - 2\lambda I_\lambda), \\ D_\lambda = \tfrac{1}{8}[-4\lambda^2 I_\lambda + 2I_\lambda' + I_\lambda''], \\ E_\lambda = \tfrac{1}{8}[-4\lambda^2 I_\lambda + 2'I_\lambda + ''I_\lambda], \\ F_\lambda = \tfrac{1}{4}[4\lambda^2 I_\lambda - 2\lambda(I_\lambda' + 'I_\lambda) + 'I_\lambda'], \\ G_\lambda = -\tfrac{1}{4}(III_{\lambda-1} + III_{\lambda+1}), \\ H_\lambda = \tfrac{1}{8}[(4\lambda^2-5\lambda)I_\lambda - (4\lambda+2)I_\lambda' + I_\lambda''], \\ K_\lambda = \tfrac{1}{4}[-4\lambda^2 I_\lambda + 2\lambda(I_\lambda' - 'I_\lambda) + 'I_\lambda'] \\ L_\lambda = \tfrac{1}{2}III_\lambda, \end{cases}$$

when for brevity the following symbols are used:

$$(4)\qquad I_\lambda=(1^\lambda),\quad I_\lambda'=a_1\frac{\partial(1^\lambda)}{\partial a_1},\quad 'I_\lambda=a_2\frac{\partial(1^\lambda)}{\partial a_2},\quad I_\lambda''=a_1^2\frac{\partial^2(1^\lambda)}{\partial a_1^2},$$

$$'I_\lambda'=a_1a_2\frac{\partial^2(1^\lambda)}{\partial a_1\partial a_2},\quad ''I_\lambda=a_2^2\frac{\partial^2(1^\lambda)}{\partial a_2^2},\quad III_\lambda=a_1a_2(3^\lambda).$$

By equation (20), §25 and the results obtained by differentiating it with respect to a_1 and a_2, the following relations may be found to exist among the coefficients.

$$I_\lambda'+'I_\lambda=-I_\lambda,\quad I_\lambda''+'I_\lambda'=-2I_\lambda',\quad 'I_\lambda'+''I_\lambda=-2'I_\lambda,$$

by the aid of which the coefficients may be further transformed. We find, for example

$$D_\lambda=E_\lambda=-\tfrac{1}{8}(4\lambda^2I_\lambda+'I_\lambda'),\qquad F_\lambda=\tfrac{1}{4}[(4\lambda^2+2\lambda)I_\lambda+'I_\lambda'].$$

Further, it follows from (12), § 25, that

$$III_{\lambda-1}+III_{\lambda+1}=k\,III_\lambda-I_\lambda,$$

and, therefore,

$$G_\lambda=-\tfrac{1}{4}(kIII_\lambda-I_\lambda).$$

If we wish to produce the entire expeditious development of the perturbing function, we must proceed as in § 24, page 180, The angle x is small, of second order, and all terms except the first can be neglected. We have here to put $\cos x=1$, $\sin x=\frac{1}{2}i_1i_2\sin(\Omega_1-\Omega_2)$, so that we obtain

$$\Sigma A_\lambda\cos\varphi=\Sigma A_\lambda\cos\psi+\tfrac{1}{2}i_1i_2\Sigma\lambda A_\lambda\cos(\psi+\Omega_1-\Omega_2),$$

where for brevity

$$(5)\qquad \lambda(\zeta_1-\zeta_2)=\psi.$$

Further, the term $zG_\lambda\cos\varphi$ separates into the three terms $\frac{1}{4}i_1^2G_\lambda\cos\psi+\frac{1}{4}i_2^2G_\lambda\cos\psi-\frac{1}{2}i_1i_2G_\lambda\cos[\psi+(\Omega_1-\Omega_2)]$,

while the last term $zL_\lambda\cos(\varphi+l_1+l_2)$ separates into

$$\tfrac{1}{4}i_1^2L_\lambda\cos(\psi+\xi_1+\zeta_2-2\Omega_1)+\tfrac{1}{4}i_2^2L_\lambda\cos(\psi+\zeta_1+\xi_2-2\Omega_2)$$
$$-\tfrac{1}{2}i_1i_2L_\lambda\cos(\psi+\zeta_1+\zeta_2-\Omega_1-\Omega_2).$$

The equation (2) now changes into

$$(6)\qquad r_{12}^{-1} = \sum_{\lambda=-\infty}^{\lambda=+\infty} [A_\lambda \cos\psi + e_1 B_\lambda \cos(\psi+\zeta_1-\pi_1)$$
$$+ e_2 C_\lambda \cos(\psi+\zeta_2-\pi_2) + e_1^2 D_\lambda \cos\psi + e_2^2 E_\lambda \cos\psi$$
$$+ e_1 e_2 F_\lambda \cos(\psi+\zeta_1-\zeta_2-\pi_1+\pi_2) + \tfrac{1}{4} i_1^2 G_\lambda \cos\psi$$
$$+ \tfrac{1}{4} i_2^2 G_\lambda \cos\psi + i_1 i_2 G_\lambda' \cos(\psi+\Omega_1-\Omega_2)$$
$$+ e_1^2 H_\lambda \cos(\psi+2\zeta_1-2\pi_1) + e_2^2 J_\lambda \cos(\psi+2\zeta_2-2\pi_2)$$
$$+ e_1 e_2 K_\lambda \cos(\psi+\zeta_1-\pi_1+\zeta_2-\pi_2)$$
$$+ \tfrac{1}{4} i_1^2 L_\lambda \cos(\psi+\zeta_1+\zeta_2-2\Omega_1)$$
$$+ \tfrac{1}{4} i_2^2 L_\lambda \cos(\psi+\zeta_1+\zeta_2-2\Omega_2)$$
$$- \tfrac{1}{2} i_1 i_2 L_\lambda \cos(\psi+\zeta_1+\zeta_2-\Omega_1-\Omega_2)].$$

The coefficients A_λ to L_λ are those given in (3),

$$G_\lambda' = -\tfrac{1}{2} G_\lambda + \tfrac{1}{2}\lambda A_\lambda = \tfrac{1}{8}(III_{\lambda-1} + III_{\lambda+1} + 2\lambda I_\lambda).$$

By means of (8) and (12) of §25, we have finally

$$(7)\qquad G_\lambda' = \tfrac{1}{4} III_{\lambda-1}.$$

This completes the entire development of the perturbing function up to terms of the second degree of the eccentricities and inclinations.

In the next section, we shall see that the terms of r_{12}^{-1} which are independent of ζ_1 and ζ_2 and therefore of the time, play an important part. We will collect them into a single term,— the so-called *secular term of the perturbing function.* If we designate this term by $\left[\frac{1}{r_{12}}\right]$, we have

$$\left[\frac{1}{r_{12}}\right] = \tfrac{1}{2} I_0 - \tfrac{1}{8} e_1^2 {}'I_0' - \tfrac{1}{8} e_2^2 {}'I_0' + \tfrac{1}{4} e_1 e_2 (2 I_1 + {}'I_1') \cos(\pi_1 - \pi_2)$$
$$- (\tfrac{1}{8} i_1^2 + \tfrac{1}{8} i_2^2 - \tfrac{1}{4} i_1 i_2 \cos(\Omega_1 - \Omega_2))\, III_1.$$

The coefficients can be somewhat transformed.

If we put $i=0$, $s=1$, then $i=1$, $s=1$, the equation (19) §25 gives

$$(8)\qquad \begin{cases} I_0' = \dfrac{-a_1^2 I_0 + a_1 a_2 I_1}{a_1^2 - a_2^2}, \\[2ex] I_1' = \dfrac{-a_1 a_2 I_0 + a_2^2 I_1}{a_1^2 - a_2^2}, \end{cases}$$

therefore

(9) $$I_0' a_2 - I_1' a_1 = 0,$$

and likewise

$$-'I_0 a_1 + 'I_1 a_2 = 0,$$

and from this it follows that, if (8) be differentiated with respect to a_2 and then multiplied by a_2,

(10) $$'I_0' = \frac{a_1 a_2 I_1}{a_1^2 - a_2^2} + \frac{I_0' 2 a_2^2}{a_1^2 - a_2^2} = \frac{a_2^2 I_0' - a_1^{2\prime} I_0}{a_1^2 - a_2^2}.$$

It likewise follows that

$$'I_1' = \frac{a_1 a_2 (I_0' - 'I_0)}{a_1^2 - a_2^2}.$$

On the other hand, if we take $s = 1$, $i = 1$ and then 0, equation (18), § 25 gives

(11) $$2I_1' = -\frac{(a_1^2 - a_2^2)}{a_1 a_2} III_1 - I_1, \quad 2I_0' = -\frac{(a_1^2 - a_2^2)}{a_1 a_2} III_0 - I_0,$$

and by the use of (8) and (9)

(12) $$'I_0' = -III_1, \quad 'I_1' = -III_0.$$

Therefore, by (8), (10) and (12)

$$2I_1 + 'I_1' = -2k\, III_1 + 3\, III_0,$$

and therefore by (10), § 25,

(13) $$2\, I_1 + 'I_1' = -III_2.$$

We, therefore, finally get

(14) $$\left[\frac{1}{r_{12}}\right] = \tfrac{1}{2} I_0 + \tfrac{1}{8} III_1 [e_1^2 + e_2^2 - i_1^2 - i_2^2 + 2 i_1 i_2 \cos(\Omega_1 - \Omega_2)]$$
$$- \tfrac{1}{4} III_2 e_1 e_2 \cos(\pi_1 - \pi_2)$$

The term $\frac{1}{2} I_0$ is entirely independent of the eccentricities and inclinations. The six other terms divide into two groups of three terms each, of which the first depend only on the eccentricities and longitudes of perihelia, the second group only on the inclinations and the longitudes of the nodes. The sum of the terms of the second group is negative, since III_1 is positive. The opposite is the case with the first group, because, by (16), § 25, an inequality, $III_2 < III_1$ exists.

Besides, the considerations of §24 show that $\left[\frac{1}{r_{12}}\right]$ is of even degree in relation to the eccentricities and inclinations, so that (14) is exact to the third degree.

We will now develop the other part of the perturbing function. The equations (19), §4, give at once

$$
\begin{aligned}
(15)\qquad x_1x_2+y_1y_2+z_1z_2 = a_1a_2[&\cos(\zeta_1-\zeta_2)\\
&+\tfrac{1}{2}e_1(-3\cos(\zeta_2-\pi_1)+\cos(2\zeta_1-\zeta_2-\pi_1))\\
&+\tfrac{1}{2}e_2(-3\cos(\zeta_1-\pi_2)+\cos(2\zeta_2-\zeta_1-\pi_2))\\
&+\tfrac{1}{8}e_1^2(3\cos(3\zeta_1-\zeta_2-2\pi_1)-4\cos(\zeta_1-\zeta_2)\\
&\qquad+\cos(2\pi_1-\zeta_1-\zeta_2))\\
&+\tfrac{1}{8}e_2^2(3\cos(3\zeta_2-\zeta_1-2\pi_2)-4\cos(\zeta_2-\zeta_1)\\
&\qquad+\cos(2\pi_2-\zeta_2-\zeta_1))\\
&+\tfrac{1}{4}e_1e_2(9\cos(\pi_1-\pi_2)-3\cos(2\zeta_1-\pi_1-\pi_2)\\
&\qquad-3\cos(2\zeta_1-\pi_1-\pi_2)+\cos(2\zeta_1-2\zeta_2-\pi_1+\pi_2))\\
&+\tfrac{1}{4}i_1^2(\cos(2\Omega_1-\zeta_1-\zeta_2)-\cos(\zeta_1-\zeta_2))\\
&+\tfrac{1}{4}i_2^2(\cos(2\Omega_2-\zeta_1-\zeta_2)-\cos(\zeta_2-\zeta_1))\\
&+\tfrac{1}{2}i_1i_2(\cos(\zeta_1-\zeta_2-\Omega_1+\Omega_2)-\cos(\zeta_1+\zeta_2-\Omega_1-\Omega_2))].
\end{aligned}
$$

From this we can at once develop $\dfrac{x_1x_2+y_1y_2+z_1z_2}{r_2^3}$ for

$$\frac{x_2}{r_2^3}=-\frac{d^2x_2}{\mu_2dt^2}=-\frac{n_2^2}{\mu_2}\frac{\partial^2x_2}{\partial\zeta_2^2}, \text{ etc.,}$$

and therefore,

$$(16)\qquad \frac{x_1x_2+y_1y_2+z_1z_2}{r_2^3}=-\frac{n_2^2}{\mu_2}\frac{\partial^2(x_1x_2+y_1y_2+z_1z_2)}{\partial\zeta_2^2}.$$

In (16) there can be no term independent of ζ_2, that is no secular term. If such a term exists in (15) it disappears by the differentiation.

27. The Analytical Expressions for the Perturbations.

In the preceding sections we have developed R in terms of the elements a, e, Ω, i, π, ζ. ζ is not properly an element, but contains the time t, since $\zeta=nt+\varepsilon$. If we replace the element ε by ζ in (15), §22, we will not change the derivatives with respect to e, Ω, i, π. Further

(1) $$\frac{\partial R}{\partial \varepsilon}=\frac{\partial R}{\partial \zeta}.$$

On the other hand $\frac{\partial R}{\partial a}$ receives a new meaning. If we denote its former value by $\left(\frac{\partial R}{\partial a}\right)$ and its new one by $\frac{\partial R}{\partial a}$, it follows at once that

(2) $$\left(\frac{\partial R}{\partial a}\right)=\frac{\partial R}{\partial a}+\frac{\partial R}{\partial \zeta}\frac{\partial \zeta}{\partial a}=\frac{\partial R}{\partial a}-\frac{3}{2}\frac{nt}{a}\frac{\partial R}{\partial \zeta}.$$

In the same manner

(3) $$\frac{\partial x}{\partial \varepsilon}=\frac{\partial x}{\partial \zeta},\quad \left(\frac{\partial x}{\partial a}\right)=\frac{\partial x}{\partial a}-\frac{3}{2}\frac{nt}{a}\frac{\partial x}{\partial \zeta},\ \text{etc.}$$

If we substitute this in (14), §22, we find by the aid of table (22), §11, that there is in F one additional term, namely,

$$+\frac{3}{a^2}\int^t t\frac{\partial R}{\partial \zeta}\,dt.$$

Further, in (4), §22, to the term

$$A\frac{\partial x}{\partial a}=\tfrac{1}{2}\sqrt{\frac{a}{\mu}}\frac{\partial x}{\partial a}\int^t\frac{\partial R}{\partial \zeta}\,dt$$

there is to be added another

$$-\frac{3t}{a^2}\frac{\partial x}{\partial \zeta}\int\frac{\partial R}{\partial \zeta}\,dt.$$

There arises, therefore, in δx the additional term

$$+\frac{3}{a^2}\frac{\partial x}{\partial \zeta}\left(\int t\frac{\partial R}{\partial \zeta}\,dt-t\int\frac{\partial R}{\partial \zeta}\,dt\right)$$
$$=-\frac{3}{a^2}\frac{\partial x}{\partial \zeta}\int\left(\int\frac{\partial R}{\partial \zeta}\,dt\right)dt.$$

There are corresponding additional terms for δy and δz. We see then that in (15) §22, we must write

$$\frac{\partial R}{\partial a}+\frac{3n}{2a}\int\frac{\partial R}{\partial \zeta}\,dt$$

instead of $\frac{\partial R}{\partial a}$.

If we designate the disturbed planet by the subscript 1 and the disturbing planet by the subscript 2, the part of R which depends on these two planets, gives rise to terms of the form

$$K \cos L, \tag{4}$$

in which the coefficient K depends only on the major axes, the excentricities and the inclinations, while L is a linear function of the mean longitudes, the longitudes of perihelia, and the longitudes of the nodes, with integral coefficients whose sum is zero. By the differentation of (4) with respect to one of the elements $a_1, e_1, \Omega_1, i_1, \pi_1\ \zeta_1$, there arises a term of the form

$$K_1 \frac{\sin}{\cos}(L). \tag{5}$$

The angle L is a linear function of the time, and is, in fact, the coefficient of t in (48), § 25, and

$$= \mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2. \tag{6}$$

By the integration of (5) with respect to t we get, if (6) is not zero, a term of the form

$$\frac{K_1}{\mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2}\left(\frac{+\sin}{-\cos}(L)\right). \tag{7}$$

The coefficient (6) becomes zero

$$(1) \text{ if } \mathfrak{a}_1 = \mathfrak{a}_2 = 0, \tag{8}$$

and, therefore, for all secular terms. Then by integration of (5) with respect to t, a term of the form

$$\left(K_1 \frac{\cos}{\sin}(L)\right) t, \tag{9}$$

is obtained, that is a term proportional to the time. And zero

$$(2) \quad \text{if} \quad n_1 : n_2 = \mathfrak{a}_2 : -\mathfrak{a}_1, \tag{10}$$

that is when the two mean daily motions, and therefore when the two periods are in commensurable ratio to each other. We have seen that there is much discretion in the selection of the original elements. We can choose the periodic times in such a way that equation (10) is never fulfilled, or fulfilled for such large entire numbers $\mathfrak{a}_1$ and $\mathfrak{a}_2$ (exact or approximate), that the degree of the term is very large, and therefore the coefficient K

very small, so that this term for very long intervals of time may be neglected. We shall see later that there is a very suitable selection of elements and that with this the periodic times of the planets are commensurable only for very large values of $\mathfrak{a}_1$ and $\mathfrak{a}_2$. We see, then, that practically, only the secular terms of the perturbing function give proportional terms by integration with respect to the time.

We will, therefore, divide the perturbing function into the so-called periodic part and the secular part and designate them by (R) and $[R]$.

In § 24, it was shown that R_1 can be developed into a sum of the form (49). (R_1) is then the sum of the terms of R_1 for which the two integrals $\mathfrak{a}_1$, $\mathfrak{a}_2$ are not both zero at the same time, while $[\mathfrak{R}_1]$ is the sum of those terms for which $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are zero together.

Let

(11) $\quad K e_1^{\alpha_1} e_2^{\alpha_2} i_1^{\beta_1} i_2^{\beta_2} \cos(\mathfrak{a}_1 \zeta_1 + \mathfrak{a}_2 \zeta_2 + \mathfrak{b}_1 \pi_1 + \mathfrak{b}_2 \pi_2 + \mathfrak{c}_1 \Omega_1 + \mathfrak{c}_2 \Omega_2)$,

or briefly

(12) $$H \cos L,$$

be a term of (R_1) where for integral numbers, the conditions of § 24 hold, that is

(13) $$\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{b}_1 + \mathfrak{b}_2 + \mathfrak{c}_1 + \mathfrak{c}_2 = 0,$$

and the four differences

$$\alpha_1 - [\mathfrak{b}_1],\ \alpha_2 - [\mathfrak{b}_2],\ \beta_1 - [\mathfrak{c}_1],\ \beta_2 - [\mathfrak{c}_2]$$

are even, positive, or zero.

Further, let

(14) $\quad k e_1^{\alpha_1'} i_1^{\beta_1'} \cos(\mathfrak{a}_1' \zeta_1 + \mathfrak{b}_1' \pi_1 + \mathfrak{c}_1' \Omega_1)$, or briefly, $H' \cos L'$

be a term of x, so that according to § 4,

$$\mathfrak{a}_1' + \mathfrak{b}_1' + \mathfrak{c}_1' = 1,$$

and

$$\alpha_1' - [\mathfrak{b}_1'],\quad \beta_1' - [\mathfrak{c}_1']$$

are even, positive or zero.

If we now denote the part of δx which depends on (R_1) by (δx_1), it follows by the use of the equations (15) and (11) of § 22, and the table (22), § 11, that

$$(15)\qquad (\delta x_1) = \Sigma\left[M_1 \cos(L' + L) + M_2 \cos(L' - L)\right],$$

in which

$$(15a)\qquad M_1 = \frac{1}{\sqrt{a_1 \mu_1}}\Bigg[-\frac{3}{2} n_1 H H' \frac{\mathfrak{a}_1 \mathfrak{a}_2'}{(\mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2)^2}$$

$$-\frac{\partial H}{\partial a_1} H' a_1 \frac{\mathfrak{a}_1'}{\mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2}$$

$$+\frac{H H'}{\mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2}\Bigg\{\mathfrak{a}_1 + \frac{\sqrt{1-e_1^2}}{2e_1^2}(-\alpha_1' \mathfrak{b}_1 + \alpha_1 \mathfrak{b}_1')$$

$$+\frac{(1-\sqrt{1-e_1^2})\sqrt{1-e_1^2}}{2e_1^2}(-\alpha_1' \mathfrak{a}_1 + \alpha_1 \mathfrak{a}_1')$$

$$+\frac{1}{2 i_1 \sin i_1 \sqrt{1-e_1^2}}(-\beta_1' \mathfrak{c}_1 + \beta_1 \mathfrak{c}_1')$$

$$+\frac{1-\cos i_1}{2 i_1 \sin i_1 \sqrt{1-e_1^2}}(-\beta_1'(\mathfrak{a}_1 + \mathfrak{b}_1) + \beta_1(\mathfrak{a}_1' + \mathfrak{b}_1'))\Bigg\}\Bigg]$$

The second term $M_2 \cos(L' - L)$ is formed from the first by exchanging the six integral numbers in L with their opposites. The symbol Σ in (15) refers not only to the integral numbers in (11) and (14) but to those of all the disturbing planets. If (15*a*) is developed in ascending powers of the eccentricities and inclinations, the law of formation is at once seen to be

$$(16)\qquad (\delta x_1) = \Sigma (K) e_1^{\alpha_1} e_2^{\alpha_2} i_1^{\beta_1} i_2^{\beta_2} \cos(\mathfrak{a}_1 \zeta_1 + \mathfrak{a}_2 \zeta_2 + \mathfrak{b}_1 \pi_1 + \mathfrak{b}_2 \pi_2 + \mathfrak{c}_1 \Omega_1 + \mathfrak{c}_2 \Omega_2),$$

where the whole numbers $\alpha_1, \alpha_2 \ldots$ fulfill exactly the same conditions as before, except that the sum (13) is not now $= 0$ but $= 1$. (δy_1) follows at once from (δx_1) by putting sine for cosine. Finally (δz_1) has the same form as (δy_1) except that for (δz_1) the sum (13) again $= 0$, while $\mathfrak{c}_1 + \mathfrak{c}_2$ is odd.

If, for brevity, we put

$$i(\zeta_1 - \zeta_2) = \psi$$

and limit ourselves to the consideration of terms of the first degree, the following terms result:

$$(17)\qquad (\delta x_1) = \Sigma\big[(K_i^1)\cos(\psi + \zeta_1) + e_1(K_i^2)\cos(\psi + \pi_1) + e_1(K_i^3)\cos(\psi + 2\zeta_1 - \pi_1) + e_2(K_i^4)\cos(\psi + \pi_2) + e_2(K_i^5)\cos(\psi + 2\zeta_2 - \pi_2),$$

$$(18)\quad (\delta y_1) = \Sigma \big[(K_i^1) \sin(\psi + \zeta_1) + e_1(K_i^2) \sin(\psi + \pi_1) + e_1(K_i^3) \sin(\psi + 2\zeta_1 - \pi_1) + e_2(K_i^4) \sin(\psi + \pi_2) + e_2(K_i^5) \sin(\psi + 2\zeta_2 - \pi_2) \big],$$

$$(19)\quad (\delta z_1) = \Sigma \big[i_1(K_i^6) \sin(\psi + \zeta_1 - \Omega_1) + i_2(K_i^7) \sin(\psi + \zeta_2 - \Omega_2) \big],$$

where (K_i) represents coefficients depending only on a_1 and a_2 and which are to be formed from what precedes. It is to be noted that, in order to take account of the perturbations up to the n^{th} degree, the perturbing function must have been developed to the $(n+1)^{\text{th}}$ degree.

We now pass to the secular part $[R_1]$ of the perturbing function and to the corresponding terms $[\delta x_1]$, $[\delta y_1]$, $[\delta z_1]$. For this part

$$\frac{\partial [R_1']}{\partial \zeta_1} = 0$$

and

$$\int \frac{\partial [R_1]}{\partial a_1} dt = t \frac{\partial [R_1]}{\partial a_1}, \text{ etc.}$$

From this it follows that

$$(20)\quad [\delta x_1] = t \Bigg[(a, \varepsilon)_1 \frac{\partial [R_1]}{\partial a_1} \frac{\partial x_1}{\partial \zeta_1} + (e, \pi)_1 \left(\frac{\partial [R_1]}{\partial e_1} \frac{\partial x_1}{\partial \pi_1} - \frac{\partial [R_1]}{\partial \pi_1} \frac{\partial x_1}{\partial e_1} \right) + (e, \varepsilon)_1 \frac{\partial [R_1]}{\partial e_1} \frac{\partial x_1}{\partial \zeta_1} + (i, \Omega)_1 \left(\frac{\partial [R_1]}{\partial i_1} \frac{\partial x_1}{\partial \Omega_1} - \frac{\partial [R_1]}{\partial \Omega_1} \frac{\partial x_1}{\partial i_1} \right) + (i, \varepsilon)_1 \frac{\partial (R_1]}{\partial i_1} \frac{\partial x_1}{\partial \zeta_1} + (i, \pi)_1 \left(\frac{\partial [R_1]}{\partial i_1} \frac{\partial x_1}{\partial \pi_1} - \frac{\partial [R_1]}{\partial \pi_1} \frac{\partial x_1}{\partial i_1} \right) \Bigg].$$

Corresponding expressions can be obtained for $[\delta y_1]$ and $[\delta z_1]$. By the development of (20) we now get

$$(21)\quad [\delta x_1] = t \Sigma [K] e_1^{\alpha_1} e_2^{\alpha_2} i_1^{\beta_1} i_2^{\beta_2} \sin(\mathfrak{a}_1 \xi_1 + \mathfrak{b}_1 \pi_1 + \mathfrak{b}_2 \pi_2 + \mathfrak{c}_1 \Omega_1 + \mathfrak{c}_2 \Omega_2).$$

The same conditions hold here for the integral numbers as in (16) except $\mathfrak{a}_2$ is always zero. $[\delta y_1]$ is formed from $[\delta x_1]$ by substituting —cosine for sine. $[\delta z_1]$ has the same form as $[\delta y_1]$, differing from it only as (δz_1) differed from (δy_1).

Finally six arbitrary constants appear in the six integrals $\int \frac{\partial R_1}{\partial a_1} dt$, etc., of equations (15), § 22. These constants will be

denoted by $\{a_1\}, \ldots$ They give another term $\{\delta x_1\}$ to δx_1, as follows:

$$\{\delta x_1\} = \frac{3n_1}{2a_1}(a, \varepsilon)_1 \{\zeta_1\} \, t \frac{\partial x_1}{\partial \zeta_1} + \frac{\partial x_1}{\partial \zeta_1}(a, \varepsilon)_1 \{a_1\} + \ldots$$

The constants can be taken entirely arbitrarily.

Laplace puts

$$\{\zeta_1\} = -\frac{2a_1}{3n_1}\frac{[\partial R_1]}{\partial a_1},$$

and finds that the term of $[\delta x] + \{\delta x\}$ which is proportional to the time and depends upon $\frac{[\partial R_1]}{\partial a_1}$ vanishes. He puts $\{a_1\}$, $\{i_1\}$ and $\{\Omega_1\} = 0$ and assigns to $\{e_1\}$ and $\{\pi_1\}$ certain definite values in order to simplify his equations to a certain degree.

We have now developed the general form of the perturbations of the first degree in reference to the disturbing masses. If we go from this to the second degree the labor increases enormously, but it is easy to see that the perturbations consist of terms of the three following forms

$$K \frac{\cos}{\sin} L, \quad Kt \frac{\cos}{\sin} L, \quad Kt^2 \frac{\cos}{\sin} L,$$

only that in this case K and L can contain the elements of three planets, one disturbed and two disturbing. In perturbations of the third degree, t^3 enters as a factor in some of the terms and the elements of four planets are involved, one disturbed, three disturbing, etc.

The calculation of these perturbations is very tedious and it is well to have a simple control of their accuracy. In order to obtain such a control, let M_1' be what (15a) becomes when the term which has $(\mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2)$ in the denominator is omitted and the denominator $(\mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2)$ is dropped from the others. By this means (δx_1) passes into $(\delta' x_1)$. Moreover, if the factor t is omitted in (21) and the sine changed into cosine, $[\delta x_1]$ passes into $[\delta' x_1]$. Then, identically,

$$(\delta' x_1) + [\delta' x_1] = 0.$$

For the first member is then the development of

$$\frac{\partial R_1}{\partial a_1}\frac{\partial a_1}{\partial x_1}+\frac{\partial R_1}{\partial e_1}\frac{\partial e_1}{\partial x_1'}+\dots$$

which expression, according to § 25 is equal to zero.

We have seen that the perturbations of the first degree of the disturbing masses, separate into two great groups. The first group consists of terms, each of which has the form

$$a \cos (bt + c),$$

in which a, b and c are constants. The second group on the other hand consists of the terms of the form

$$at \cos (bt + c).$$

When this development of the coordinates and of the perturbing functions converges, which is the case for the elements of our system, the developments of (δx) and $[\delta x]$ will also converge. The latter, however, in spite of the smallness of the masses increases with the time beyond reasonable limit by reason of the time entering as a factor, and the same thing is true of perturbations of the higher degrees. From this it is evident that the convergence obtains only for limited durations which may be tens or even hundreds of years, and that the increase in the perturbations is to be attributed chiefly to the secular term.

By the appearance of the factor t, the earlier mathematicians, especially Euler, Laplace and Lagrange, were led to some remarkable and, to us, fanciful speculations, the object of which was to overcome the difficulties presented. See *Mécanique céleste*, Tome I, p. 266. These attempts were the forerunners of a second great theory, *The Theory of the Variation of Elements*, which we shall develop in the following sections.

28. The Variation of Elements.

The fundamental ideas of the theory now to be developed are very simple and clear. At any given instant the planets all have definite coordinates and definite velocities about the sun considered as fixed. These would belong to invariable ellipses if the sun alone attracted them and if they exercised no attraction on it nor on each other, and the elements of these ellipses would

depend on the coordinates and velocities in the manner given in § 2. Whatever instant selected, the same values for the elements would always result.

The actual state of the case is, that we get different elements for different instants. However, the masses of the planets are so small, that for considerable periods, they move nearly in accordance with Kepler's laws. In fact the accordance is so great that Kepler, by ingenious combinations, deduced his laws directly from observations. The observations, which are themselves not rigorously exact, would only gradually show that the ellipses themselves are slowly changing, or, in other words, that the elements are slowly changing. This relatively slow change goes on continuously. If, from the instantaneous coordinates and velocities, we determine for any instant the ellipse which the planet would describe from that time on if the mutual action of the planets were to cease, we find that the planet actually moves in this ellipse only for an instant. At the next instant it is moving in another ellipse nearly like it. The actual path of the planet is the curve which touches all the instantaneous ellipses, that is, it is their envelope.

This view of the planetary orbits is now to be expressed in the language of analysis. The elements of the planets will be taken as functions of the time which are to be more exactly determined. When the determination has been made, the formulas of §§ 1–4 will apply as before and the coordinates and velocities may be expressed in terms of the elements a, e, i, Ω, π, ε, and the time t. They and all the quanities depending upon them will contain the time in two different ways: first, explicitly, as it appears in the mean longitude, and second, implicitly, for the elements are functions of the time.

We have now to ascertain how the elements may be determined as functions of the time. If, for example, x is to be expressed in terms of the elements and the time as in § 4, we get

$$(1) \qquad * \qquad \frac{dx}{dt} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial a}\frac{da}{dt} + \frac{\partial x}{\partial e}\frac{de}{dt} + \dots$$

$$= \frac{\partial x}{\partial \varepsilon} n + \frac{\partial x}{\partial a}\frac{da}{dt} + \frac{\partial x}{\partial e}\frac{de}{dt} + \dots$$

The component velocities also contain the time as before, that is

(2) * $$\frac{dx}{dt} = \frac{\partial x}{\partial t} = x' = \frac{\partial x}{\partial \varepsilon} n,$$

From (1) and (2) it follows that

(3) $$\begin{cases} 0 = \dfrac{\partial x}{\partial a}\dfrac{da}{dt} + \dfrac{\partial x}{\partial e}\dfrac{de}{dt} + \ldots \\ 0 = \dfrac{\partial y}{\partial a}\dfrac{da}{dt} + \dfrac{\partial y}{\partial e}\dfrac{de}{dt} + \ldots \\ 0 = \dfrac{\partial z}{\partial a}\dfrac{da}{dt} + \dfrac{\partial z}{\partial e}\dfrac{de}{dt} + \ldots \end{cases}$$

The second drivatives

* $$\frac{d^2x}{dt^2} = \frac{\partial^2 x}{\partial t^2} + \frac{\partial x'}{\partial a}\frac{da}{dt} + \frac{\partial x'}{\partial e}\frac{de}{dt} + \ldots$$

are obtained by differentiating (2), and on substitution in the equations

(4) * $$\frac{d^2x}{dt^2} = -\frac{\mu x}{r^3} + \frac{\partial R}{\partial x}$$

and by the use of

* $$\frac{\partial^2 x}{\partial t^2} = -\frac{\mu x}{r^3},$$

become

(5) $$\begin{cases} \dfrac{\partial x'}{da}\dfrac{da}{dt} + \dfrac{\partial x'}{\partial e}\dfrac{de}{dt} + \ldots = \dfrac{\partial R}{\partial x}, \\ \dfrac{\partial y'}{\partial a}\dfrac{da}{dt} + \dfrac{\partial y'}{\partial e}\dfrac{de}{dt} + \ldots = \dfrac{\partial R}{\partial y}, \\ \dfrac{\partial z'}{\partial a}\dfrac{da}{dt} + \dfrac{\partial z'}{\partial e}\dfrac{de}{dt} + \ldots = \dfrac{\partial R}{\partial z}. \end{cases}$$

After the coefficients $\frac{da}{dt}, \frac{de}{dt} \ldots$ and $\frac{\partial R}{\partial x}, \ldots$ have been expressed in terms of the elements and the time, equations (3) and (5) constitute six differential equations of the first order and form a peculiar analytical transformation of the original system (4). Equations (5) are of course to be formed for all the planets. By the aid of his symbols Lagrange succeeded in transforming these equations so as to enable them to be more

easily treated. If equations (5) are multiplied in order by $\frac{\partial x}{\partial a}, \frac{\partial y}{\partial a}, \frac{\partial z}{\partial a}$, and equations (3) by $-\frac{\partial x'}{\partial a}, -\frac{\partial y'}{\partial a}, -\frac{\partial z'}{\partial a}$, we get by addition

$$(6)\quad \begin{cases} [a,a]\frac{da}{dt}+[a,e]\frac{de}{dt}+[a,i]\frac{di}{dt}+\ldots=\frac{\partial R}{\partial a}, \\ \text{and similarly,} \\ [e,a]\frac{da}{dt}+[e,e]\frac{de}{dt}+[e,i]\frac{di}{dt}+\ldots=\frac{\partial R}{\partial e}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

The equations (6) are to be solved for $\frac{da}{dt}, \frac{de}{dt}, \ldots$ According to §10, these solutions are

$$(7)\quad \begin{cases} \frac{da}{dt}=(a,a)\frac{\partial R}{\partial a}+(e,a)\frac{\partial R}{\partial e}+(\varepsilon,a)\frac{\partial R}{\partial \varepsilon}+\ldots \\ \frac{de}{dt}=(a,e)\frac{\partial R}{\partial a}+(e,e)\frac{\partial R}{\partial e}+(\varepsilon,e)\frac{\partial R}{\partial \varepsilon}+\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

and by the use of (21) and (22), §11,

$$(8)\quad \begin{cases} \frac{da}{dt}=2\sqrt{\frac{a}{\mu}}\frac{\partial R}{\partial \varepsilon}, \\ \frac{de}{dt}=-\frac{\sqrt{1-e^2}(1-\sqrt{1-e^2})}{e\sqrt{a\mu}}\frac{\partial R}{\partial \varepsilon}-\frac{1}{\varepsilon}\sqrt{\frac{1-e^2}{a\mu}}\frac{\partial R}{\partial \pi}, \\ \frac{d\varepsilon}{dt}=-2\sqrt{\frac{a}{\mu}}\left(\frac{\partial R}{\partial a}\right)+\frac{\sqrt{1-e^2}(1-\sqrt{1-e^2})}{e\sqrt{a\mu}}\frac{\partial R}{\partial e} \\ \qquad\qquad +\frac{1-\cos i}{\sin i\sqrt{\mu a(1-e^2)}}\frac{\partial R}{\partial i}, \\ \frac{d\pi}{dt}=\frac{1}{e}\sqrt{\frac{1-e^2}{a\mu}}\frac{\partial R}{\partial e}+\frac{1-\cos i}{\sin i\sqrt{\mu a(1-e^2)}}\frac{\partial R}{\partial i}, \\ \frac{di}{dt}=-\frac{1-\cos i}{\sin i\sqrt{\mu a(1-e^2)}}\frac{\partial R}{\partial \varepsilon}-\frac{1-\cos i}{\sin i\sqrt{\mu a(1-e^2)}}\frac{\partial R}{\partial \pi} \\ \qquad\qquad -\frac{1}{\sin i\sqrt{\mu a(1-e^2)}}\frac{\partial R}{\partial \Omega}, \\ \frac{d\Omega}{dt}=\frac{1}{\sin i\sqrt{\mu a(1-e^2)}}\frac{\partial R}{\partial i}. \end{cases}$$

These equations (8) form the foundation of the theory of the variation of elements. According to Poisson's method they can be obtained easily and directly. Laplace obtained them in a more difficult way and gave them in a supplement to the second volume of *Mécanique céleste.* They have the advantages that the second members depend only on the perturbing functions and, consequently, are relatively small, that the differentation is with respect to the elements, and that the coefficients do not explicitly contain the time.

Still simpler equations could be obtained by using the canonical constants introduced in (7), § 13. The corresponding equations are

$$\frac{d\alpha_1}{dt}=-\frac{\partial R}{\partial\beta_1},\quad \frac{d\alpha_2}{dt}=-\frac{\partial R}{\partial\beta_2},\quad \frac{d\alpha_3}{dt}=-\frac{\partial R}{\partial\beta_3},$$
$$\frac{d\beta_1}{dt}=+\frac{\partial R}{\partial\alpha_1},\quad \frac{d\beta_2}{dt}=+\frac{\partial R}{\partial\alpha_2},\quad \frac{d\beta_3}{dt}=+\frac{\partial R}{\partial\alpha_3},$$

But since R has been developed in ascending powers of e and i, the equations (8) are more convenient for us and we will use them. The derivative of R with respect to a is put in parentheses because it is to be taken as complete, that is, both in so far as a is explicitly contained in R and also in the relation

$$\zeta=nt+\varepsilon. \tag{9}$$

If, as was done by Tisserand, (*Exposition, d'après les principes de Jacobi, de la méthode suivi par Delaunay—Journal de Mathématiques pure et appliquées,* 1868), we introduce the mean longitude instead of ε, we avoid the complete derivatives with respect to a and thus the explicit appearance of the time. We have

$$\frac{\partial R}{\partial\varepsilon}=\frac{\partial R}{\partial\zeta},$$

$$\left(\frac{\partial R}{\partial a}\right)=\frac{\partial R}{\partial a}-\frac{3}{2}\sqrt{\frac{\mu}{a^5}}\,t\,\frac{\partial R}{\partial\zeta}.$$

By the differentiation of (9) we also get

$$\frac{d\zeta}{dt}=n-\frac{3}{2}t\sqrt{\frac{\mu}{a^5}}\frac{da}{dt}+\frac{d\varepsilon}{dt}$$

$$=-2\sqrt{\frac{a}{\mu}}\frac{\partial\left(\frac{\mu}{2a}\right)}{\partial a}\frac{3}{2}t\sqrt{\frac{\mu}{a^5}}\frac{da}{dt}+\frac{d\varepsilon}{dt}.$$

If we also put

(10) $$R' = R + \frac{\mu}{2a},$$

we get

(11) $$\frac{d\zeta}{dt} = -2\sqrt{\frac{a}{\mu}}\frac{\partial R'}{\partial a} + \sqrt{\frac{1-e^2}{a\mu}}\frac{1-\sqrt{1-e^2}}{e}\frac{\partial R'}{\partial e} + \frac{1-\cos i}{\sin i\sqrt{\mu a(1-e^2)}}\frac{\partial R'}{\partial i}.$$

R' can also be put in the place of R in the other equations (8). Equation (11) has the advantage that it gives at once the derivatives of the mean longitude with respect to the time.

Another circumstance to be mentioned is that $\frac{d\pi}{dt}$ and $\frac{d\Omega}{dt}$ become infinite when e or $i = 0$. It is evident that when the eccentricities or inclinations are very small, small changes in the orbit cause great variations of the perihelion or node.

To overcome this difficulty Lagrange substituted for e, i, π, Ω, four new variables by means of the equations

(12) $$\begin{cases} h = e\sin\pi, & p = \sin i \sin\Omega, \\ e = e\cos\pi, & q = \sin i\cos\Omega. \end{cases}$$

From these

$$\frac{dh}{dt} = \frac{de}{dt}\frac{h}{e} + \frac{d\pi}{dt}h,$$

$$\frac{dl}{dt} = \frac{de}{dt}\frac{l}{e} - \frac{d\pi}{dt}l,$$

$$\frac{dp}{dt} = p\cot i\frac{di}{dt} + \frac{d\Omega}{dt}q,$$

$$\frac{dq}{dt} = q\cot i\frac{di}{dt} - \frac{d\Omega}{dt}p,$$

and further

$$\frac{\partial R}{\partial e} = \frac{\partial R}{\partial h}\frac{h}{e} + \frac{\partial R}{\partial l}\frac{l}{e},$$

$$\frac{\partial R}{\partial \pi} = \frac{\partial R}{\partial h}l - \frac{\partial R}{\partial l}h,$$

$$\frac{\partial R}{\partial i} = \left(p\frac{\partial R}{\partial p} + q\frac{\partial R}{\partial q}\right)\cot i,$$

$$\frac{\partial R}{\partial \Omega} = \frac{\partial R}{\partial p}q - \frac{\partial R}{\partial q}p.$$

By introducing (12) into (8), we get a less synoptic form, but there is the advantage that with vanishing eccentricities and inclinations no infinite coefficients appear. The coefficients can be developed in powers of h, l, p, q and the form holds for the perturbing function R.

The substitutions (12) will be applied later in a case limited to the second degree of the inclinations and eccentricities.

29. Approximate Integration of the Differential Equations for the Variation of the Elements.

The differential equations (8) of the preceding section are special forms of the original system (2), § 21, and it is evident that in this case also the actual integration is impossible. Yet, for two reasons, equations (8) possess a very great advantage when it is a question of approximate integration. The first is the smallness of R and the second, the analytical form to which it is reduced.

For example, from the first of equations (8), it follows that

$$a = \int^{t} 2\sqrt{\frac{a}{\mu}}\frac{\partial R}{\partial \zeta}\,dt. \tag{1}$$

But we have not yet gained anything, for in order to complete the integral, the elements must have been already determined in terms of the time, or the problem must have been already solved. In (1) it is as if with the equation $x^n = x + a$, the unknown quantity x were to be obtained by the formula $x = \sqrt[n]{x+a}$. And exactly as this formula can be used to find x by a process of approximation, beginning with a known approximate value, can formula (1) be used with the skillful employment of the development of R.

It has already been shown that R separates into a periodic part (R) and a secular part $[R]$. Let

$$k \begin{matrix}\cos\\ \sin\end{matrix}\left\{ i_1\zeta_1 + i_2\zeta_2 \right\} \tag{2}$$

be any periodic term of R_1. (We will from this on use subscripts to distinguish the disturbing from the disturbed planet.) According to (8), § 28, this term produces a term of the same

form (2) in every derivative of the elements, and hence in the elements themselves terms of the form

$$(3) \qquad \int k \frac{\cos}{\sin} \left\{ i_1 \zeta_1 + i_2 \zeta_2 \right\} dt.$$

If the elements were not variable this could be integrated at once and would give

$$(4) \qquad \frac{k}{i_1 n_1 + i_2 n_2} \frac{\sin}{-\cos} \left\{ i_1 \zeta_1 + i_2 \zeta_2 \right\}.$$

It can be shown, however, that since the disturbing masses are small, (4) is very approximately the integral of (3). For

$$\int k \frac{\cos}{\sin} (i_1 \zeta_1 + i_2 \zeta_2)\, dt$$

$$= \int \frac{k}{i_1 n_1 + i_2 n_2} \frac{\cos}{\sin} \{ i_1 \zeta_1 + i_2 \zeta_2 \} d \{ i_1 \zeta_1 + i_2 \zeta_2 \}$$

$$- \int k \frac{\cos}{\sin} \{ i_1 \zeta_1 + i_2 \zeta_2 \} \left[\frac{d (i_1 \zeta_1 + i_2 \zeta_2)}{(i_1 n_1 + i_2 n_2) dt} - 1 \right] dt.$$

The second term of the second member is an integral which, according to (11), § 28, depends only on the second powers of the disturbing masses, since k as well as $\left[\frac{d (i_1 \zeta_1 + i_2 \zeta_2)}{(i_1 n_1 + i_2 n_2)\, dt} - 1 \right]$ is of the first order with reference to those masses. Neglecting, therefore, the second powers of the disturbing masses and integrating by parts, we get

$$\int k \frac{\cos}{\sin} (i_1 \zeta_1 + i_2 \zeta_2)\, dt = \int \frac{k}{i_1 n_1 + i_2 n_2} d \left(\frac{\sin}{-\cos} \{ i_1 \zeta_1 + i_2 n_2 \} \right)$$

$$= \frac{k}{i_1 n_1 + i_2 n_2} \frac{\sin}{-\cos} \{ i_1 \zeta_1 + i_2 \zeta_2 \}$$

$$- \int \frac{\sin}{-\cos} (i_1 \zeta_1 + i_2 \zeta_2) \frac{d \left(\frac{k}{i_1 n_1 + i_2 n_2} \right)}{dt} dt.$$

But $\dfrac{d \left(\frac{k}{i_1 n_1 + i_2 n_2} \right)}{dt}$ is also of the second order with reference to the disturbing masses. Neglecting the integral in the second member this passes into (4). It appears, therefore, that a periodic term of the perturbing function produces perturbations in

the elements which depend on the same argument $(i_1\zeta_1+i_2\zeta_2)$. The coefficients $\frac{k}{i_1 n_1+i_2 n_2}$ are proportional to the disturbing masses and, therefore, do not in general reach any value of consequence. Hence

A periodic term in R alternately increases and decreases the elements but does not permanently change them.

The secular term $[R]$ has a different action. It produces similar terms in the derivatives of the elements and if the elements were constant, it would produce terms proportional to the term containing the time t. As the elements are variable this does not actually occur, but it is easy to see that the action of the secular term goes far in time and may result in an entire change in the elements.

The approximate integration of (8), § 28, may be made in the following manner. Limit R to its secular term $[R]$ and substitute this is (8). Integrate the resulting simplified differential equations. The resulting values will not be the true ones, that is, those actually existing at any moment, but the so-called *secular values of the elements.* The actual elements may be obtained by adding to the secular elements the periodic terms

$$\frac{k}{i_1 n_1+i_2 n_2} \begin{matrix}\sin\\ -\cos\end{matrix} \left\{ i_1\zeta_1+i_2\zeta_2 \right\},$$

in which the secular values may be used. The elements are now complete and the coordinates may be computed from them.

This process is not rigorous, but the error for long periods is so small that it gives a very close approximation.

30. The Secular Values of the Elements. Development of the Rigorous Equations Between Them.

In § 26, it was shown that the secular part of the perturbing functions $R_1, R_2, \ldots$ depends only on the *one quantity*

$$V=\sum\frac{m_\lambda m_\mu}{r_{\lambda\mu}}. \tag{1}$$

If we denote this by W and replace the elements by their secular values, it follows from (8), § 28, since

$$(2) \qquad \frac{\partial W}{\partial \zeta_1} = \frac{\partial W}{\partial \zeta_2} = 0,$$

that

$$(3)\begin{cases} m_1 \dfrac{da_1}{dt} = 0, \\ m_1 \dfrac{d\zeta_1}{dt} = m_1 n_1 - 2\sqrt{\dfrac{a_1}{\mu_1}}\dfrac{\partial W}{\partial a_1} + \sqrt{\dfrac{1-e_1^2}{\mu_1 a_1}}\dfrac{(1-\sqrt{1-e_1^2})}{e_1}\dfrac{\partial W}{\partial e_1} \\ \qquad\qquad + \dfrac{1-\cos i_1}{\sin i_1\sqrt{\mu_1 a_1(1-e_1^2)}}\dfrac{\partial W}{\partial i_1}, \\ m_1 \dfrac{de_1}{dt} = -\dfrac{1}{e_1}\sqrt{\dfrac{1-e_1^2}{\mu_1 a_1}}\dfrac{\partial W}{\partial \pi_1}, \\ m_1 \dfrac{d\pi_1}{dt} = \dfrac{1-\cos i_1}{\sin i_1\sqrt{\mu_1 a_1(1-e_1^2)}}\dfrac{\partial W}{\partial i_1} + \dfrac{1}{e_1}\sqrt{\dfrac{1-e_1^2}{\mu_1 a_1}}\dfrac{\partial W}{\partial e_1} \\ m_1 \dfrac{di_1}{dt} = -\dfrac{1-\cos i_1}{\sin i_1\sqrt{\mu_1 a_1(1-e_1^2)}}\dfrac{\partial W}{\partial \pi_1} - \dfrac{1}{\sin i_1\sqrt{\mu_1 a_1(1-e_1^2)}}\dfrac{\partial W}{\partial \Omega_1}, \\ m_1 \dfrac{d\Omega_1}{dt} = \dfrac{1}{\sin i_1\sqrt{\mu_1 a_1(1-e_1^2)}}\dfrac{\partial W}{\partial i_1}. \end{cases}$$

These equations are to be formed for every planet and from them the elements (or rather their secular values) are to be determined. Since ζ is not contained in W, the second of equations (3) is to be dropped and after the elements a, e, π, i, Ω have been found, the mean longitude ζ is to be determined by pure quadrature.

The first of the equations (3) is at once integrable, and gives

$$(4) \qquad a_1 = \text{constant},$$

and therefore

The major axis of each planet, aside from periodic changes, remains invariable.

This important result was first obtained by Laplace, but only as an approximation, since he neglected the higher powers of the inclinations and eccentricities. Lagrange succeeded in perfecting the theory,—by the stroke of a pen, says Jacobi,—by proving this leading proposition of the whole theory of perturbations in all its generality.

From the last four of equations (3), it follows that

$$\frac{\partial W}{\partial e_1}\frac{de_1}{dt}+\frac{\partial W}{\partial \pi_1}\frac{d\pi_1}{dt}+\frac{\partial W}{\partial i_1}\frac{di_1}{dt}+\frac{\partial W}{\partial \Omega_1}\frac{d\Omega_1}{dt}=0.$$

In the same manner, the variations which W undergoes by reason of the changes in the elements of the other planets are zero. Hence

$$(5)\qquad \begin{cases} \dfrac{dW}{dt}=0, \\ W=\text{constant.} \end{cases}$$

Further, from (3)

$$\frac{d(m_1\sqrt{\mu_1 a_1(1-e_1^2)}\cos i_1)}{dt}=\frac{\partial W}{\partial \pi_1}+\frac{\partial W}{\partial \Omega_1}.$$

Now, by § 24,

$$\frac{\partial W}{\partial \pi_1}+\frac{\partial W}{\partial \Omega_1}+\frac{\partial W}{\partial \pi_2}+\frac{\partial W}{\partial \Omega_2}+\ldots=0.$$

and hence, by adding the above equation to its corresponding one and integrating

$$(6)\quad m_1\sqrt{\mu_1 a_1(1-e_1^2)}\cos i_1+m_2\sqrt{\mu_2 a_2(1-e_2^2)}\cos i_2+\ldots=c_1,$$

that is, the sum of the areas for the xy plane is constant. Since this may be selected arbitrarily, it is true for any other plane, as the xz plane and the yz plane, and we have

$$(7)\quad m_1\sqrt{\mu_1 a_1(1-e_1^2)}\sin i_1\cos\Omega_1+m_2\sqrt{\mu_2 a_2(1-e_2^2)}\sin i_2\cos\Omega_2 +\ldots=c_2,$$

$$(8)\quad m_1\sqrt{\mu_1 a_1(1-e_1^2)}\sin i_1\sin\Omega_1+m_2\sqrt{\mu_2 a_2(1-e_2^2)}\sin i_2\sin\Omega_2 +\ldots=c_3.$$

Equations (5), (6), (7), (8) are the only exact integrals known between the secular values of the elements, not including the stability of the major axis. The last three become especially interesting when only two planets are considered. From them, by elimination, is obtained

$$(9)\qquad 0=\begin{vmatrix} c_1, & \cos i_1, & \cos i_2 \\ c_2, & \sin i_1\ \cos\Omega_1, & \sin i_2\cos\Omega_2 \\ c_3, & \sin i_1\ \sin\Omega_1, & \sin i_1\sin\Omega_2 \end{vmatrix}.$$

If the invariable plane is taken as the plane of xy, then $c_2=c_3=0$, and (9) gives

$$0=\sin(\Omega_1-\Omega_2), \text{ and therefore } \Omega_1=\Omega_2,$$

that is, the orbital planes cut the invariable plane constantly in the same straight line, or in other words, the common node of the two planes moves on the fixed invariable plane. From (7) and (8) we also get

$$(10) \qquad m_1\sqrt{\mu_1 a_1(1-e_1^2)}\sin i_1+m_2\sqrt{\mu_2 a_2(1-e_2^2)}\sin i_2=0,$$

and we can obtain i_1 and i_2 from (6) and (10) as soon as e_1 and e_2 are known.

The other exact integrals of (3) are unknown. Approximate ones must, therefore, be obtained and this will be attempted in the next section.

31. Approximate Calculation of the Secular Values of the Elements.

If the discussion of the secular part W of the perturbing function is limited to terms up to the second degree of the eccentricities and inclinations, (14), § 26 gives

$$(1) \qquad W=\Sigma\, m_\lambda m_\mu[\tfrac{1}{2}I_0+\tfrac{1}{8}III_1(e_\lambda^2+e_\mu^2-i_\lambda^2-i_\mu^2 + 2i_\lambda i_\mu\cos(\Omega_\lambda-\Omega_\mu))-\tfrac{1}{4}III_2 e_\lambda e_\mu\cos(\pi_\lambda-\pi_\mu)],$$

or, by introducing the substitutions (12), § 28,

$$(2) \qquad W=\Sigma\, m_\lambda m_\mu[\tfrac{1}{2}I_0+\tfrac{1}{8}III_1(h_\lambda^2+h_\mu^2+l_\lambda^2+l_\mu^2 -(p_\lambda-p_\mu)^2-(q_\lambda-q_\mu)^2)-\tfrac{1}{4}III_2\,(h_\lambda h_\mu+l_\lambda l_\mu)].$$

When terms of the third and higher orders are neglected, the differential equations of the preceding section become

$$(3) \qquad \left\{\begin{aligned} m_\lambda\frac{dh_\lambda}{dt}&=\frac{1}{\sqrt{\mu_\lambda a_\lambda}}\frac{\partial W}{\partial l_\lambda},\\ m_\lambda\frac{dl_\lambda}{dt}&=-\frac{1}{\sqrt{\mu_\lambda a_\lambda}}\frac{\partial W}{\partial h_\lambda},\\ m_\lambda\frac{dp_\lambda}{dt}&=\frac{1}{\sqrt{\mu_\lambda a_\lambda}}\frac{\partial W}{\partial q_\lambda},\\ m_\lambda\frac{dq_\lambda}{dt}&=-\frac{1}{\sqrt{\mu_\lambda a_\lambda}}\frac{\partial W}{\partial p_\lambda},\end{aligned}\right\} \quad (\lambda=1,\dots n),$$

The quantity $\sqrt{\mu_\lambda a_\lambda}$ is positive for every value of λ, for all

the planets move about the sun in a positive direction with reference to the xy plane. Substituting for W its expression (2), it is easy to see that the differential equations for h and l separate completely from those for p and q, that is,

The secular variations of the eccentricities and perihelia are independent of those of the inclinations and nodes.

We pass, therefore, to the calculation of the secular values of the eccentricities and perihelia. We can limit ourselves to the terms in W which are dependent on h and l. They separate also into two groups,—those containing h only and those containing l only. These groups are

$$(4) \qquad V_1 = \Sigma m_\lambda m_\mu [\tfrac{1}{8} III_1 (h_\lambda^2 + h_\mu^2) - \tfrac{1}{4} III_2 h_\lambda h_\mu],$$

$$(5) \qquad V_2 = \Sigma m_\lambda m_\mu [\tfrac{1}{8} III_1 (l_\lambda^2 + l_\mu^2) - \tfrac{1}{4} III_2 l_\lambda l_\mu].$$

V_1 and V_2 are homogeneous functions of the second degree in h and l respectively and they are essentially positive, for by § 26, page 195,

$$(6) \qquad III_1 > III_2.$$

The substitution of (4) and (5) in the first two of equations (3), gives

$$(7) \qquad \begin{cases} m_\lambda \dfrac{dh_\lambda}{dt} = \dfrac{1}{\sqrt{\mu_\lambda a_\lambda}} \dfrac{\partial W}{\partial l_\lambda} = \dfrac{1}{\sqrt{\mu_\lambda a_\lambda}} \dfrac{\partial V_2}{\partial l_\lambda}, \\[2ex] m_\lambda \dfrac{dl_\lambda}{dt} = -\dfrac{1}{\sqrt{\mu_\lambda a_\lambda}} \dfrac{\partial W}{\partial h_\lambda} = -\dfrac{1}{\sqrt{\mu_\lambda a_\lambda}} \dfrac{\partial V_1}{\partial h_\lambda}. \end{cases}$$

The second members of these equations are, by (4) and (5), homogeneous linear functions of l and h respectively. Hence equations (7), in all $2n$ in number, are simultaneous linear differential equations with constant coefficients. By the aid of exponentials, their integration can be at once effected by known methods. To put this integration into the most elegant form, it is convenient to change (7) into a canonical system. This may be done by the aid of the substitutions

$$(8) \qquad h_\lambda \sqrt{m_\lambda \sqrt{\mu_\lambda a_\lambda}} = H_\lambda, \quad l_\lambda \sqrt{m_\lambda \sqrt{\mu_\lambda a_\lambda}} = L_\lambda,$$

which are real since $\sqrt{\mu_\lambda a_\lambda}$ is positive. Equations (7) then then become

$$(9)\qquad \frac{dH_\lambda}{dt}=\frac{\partial W}{\partial L_\lambda},\quad \frac{dL_\lambda}{dt}=-\frac{\partial W}{\partial H_\lambda},$$

in which H and L are to be introduced into W and also in V. If for brevity, we put

$$(10)\qquad \begin{cases} (\lambda,\mu)=\frac{1}{4}III_1(a_\lambda,a_\mu)\dfrac{m_\mu}{\sqrt{\mu_\lambda a_\lambda}}, \\ [\lambda,\mu]=-\frac{1}{4}III_2(a_\lambda,a_\mu)\dfrac{\sqrt{m_\lambda m_\mu}}{\sqrt[4]{\mu_\lambda a_\lambda \mu_\mu a_\mu}}, \end{cases}$$

so that

$$(11)\qquad [\lambda,\mu]=[\mu,\lambda],\quad (\lambda,\mu)m_\lambda\sqrt{\mu_\lambda a_\lambda}=(\mu,\lambda)m_\mu\sqrt{\mu_\mu a_\mu},$$

and further putting

$$(12)\qquad (\lambda,1)+(\lambda,2)+(\lambda,3)+\ldots+(\lambda,n)=[\lambda,\lambda],$$

we get

$$(13)\qquad \begin{aligned} V_1&=\tfrac{1}{2}([1,1]H_1^2+[2,2]H_2^2+[3,3]H_3^2+\ldots[n,n]H_n^2 \\ &\qquad +2[1,2,]H_1H_2+2[1,3]H_1H_3+\ldots) \\ &=\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=n}\sum_{\lambda=1}^{\mu=n}[\lambda,\mu]H_\lambda H_\mu, \end{aligned}$$

and likewise,

$$(14)\qquad V_2=\tfrac{1}{2}\sum_{\mu=1}^{\lambda=n}\sum_{\lambda=1}^{\mu=n}[\lambda,\mu]L_\lambda L_\mu.$$

We will now introduce the quadratic function

$$(15)\qquad \begin{aligned} \varphi(x_1,x_2,\ldots x_n)&=[1,1]x_1^2+[2,2]x_2^2+\ldots+2[1,2]x_1x_2+\ldots \\ &=\Sigma\Sigma[\lambda,\mu]x_\lambda x_\mu. \end{aligned}$$

Then

$$(15a)\qquad \begin{cases} V_1=\frac{1}{2}\varphi(H_1,H_2,\ldots H_n), \\ V_2=\frac{1}{2}\varphi(L_1,L_2,\ldots L_n). \end{cases}$$

The further treatment of the problem depends, as we shall hereafter see, on the transformation of the quadratic function φ into a sum of squares by means of an orthogonal substitution — a transformation which has engaged the attention of many celebrated mathematitions; Cauchy, Jacobi, Hesse, Kummer,

Borchardt, etc., being among the number. It is proper for us to recall the chief results which they have obtained.

Let the orthogonal substitution be

$$(16)\qquad \begin{cases} x_1 = a_{1,1}\, y_1 + a_{1,2}\, y_2 + a_{1,3}\, y_3 + \ldots + a_{1,n}\, y_n, \\ x_2 = a_{2,1}\, y_1 + a_{2,2}\, y_2 + a_{2,3}\, y_3 + \ldots + a_{2,n}\, y_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

by the help of which the given function of x in (15) is changed into

$$(17)\qquad \varphi(x_1, x_2, \ldots x_n) = g_1 y_1^2 + g_2 y_2^2 + \ldots + g_n y_n^2.$$

Since (16) is an orthogonal substitution, the following equations must be fulfilled

$$(18)\qquad \begin{cases} \displaystyle\sum_{\lambda=1}^{\lambda=n} a_{\lambda,\mu} a_{\lambda,\mu_1} = \begin{cases} 1 & (\mu = \mu_1) \\ 0 & (\mu \text{ not} = \mu_1), \end{cases} \\ \text{from which follows} \\ \displaystyle\sum_{\mu=1}^{\mu=n} a_{\lambda,\mu} a_{\lambda_1,\mu} = \begin{cases} 1 & (\lambda = \lambda_1) \\ 0 & (\lambda \text{ not} = \lambda_1). \end{cases} \end{cases}$$

The solutions of (16) are

$$(19)\qquad \begin{cases} y_1 = a_{1,1}\, x_1 + a_{2,1}\, x_2 + a_{3,1}\, x_3 + \ldots \\ y_2 = a_{1,2}\, x_1 + a_{2,2}\, x_2 + a_{3,2}\, x_3 + \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{cases}$$

The substitution of (19) in (17) makes the latter identical and by equating the coefficients in the two members, a sufficient number of equations of condition can be obtained to furnish the quantities a and g. In order, however, to form these conveniently, after equation (17) has been made identical, differentiate it with respect to x_1. It follows that

$$\begin{aligned} \varphi'(x_1) &= 2 g_1 y_1 \frac{\partial y_1}{\partial x_1} + g_2 y_2 + \frac{\partial y_2}{\partial x_1} + \ldots \\ &= 2 g_1 y_1 a_{1,1} + 2 g_2 y_2 a_{1,2} + \ldots \end{aligned}$$

If for $y_1, y_2, y_3, \ldots$ we select the special values, 1, 0, 0, 0, ..., we get for $x_1, x_2, x_3, \ldots$ by (16) the special values $a_{1,1}\, a_{2,1}\, a_{3,1}, \ldots$, and the preceding equation becomes

$$(20)\begin{cases} 0=\alpha_{1,1}([1,1]-g_1)+\alpha_{2,1}[2,1]+\alpha_{3,1}[3,1]+\ldots+\alpha_{n,1}[n,1], \\ \text{and likewise,} \\ 0=\alpha_{1,1}[1,2]+\alpha_{2,1}([2,2]-g_1)+\alpha_{3,1}[3,2]+\ldots \\ 0=\alpha_{1,1}[1,3]+\alpha_{2,1}[2,3]-\alpha_{3,1}([3,3]-g_1)+\ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

If we eliminate from these equations, the unknowns $\alpha_{1,1}, \alpha_{2,1}, \alpha_{3,1}, \ldots,$ we get the following final equation for g_1,

$$(21)\qquad \begin{vmatrix} [1,1]-g, & [2,1], & [3,1], & \ldots \\ [1,2], & [2,2]-g, & [3,2], & \ldots \\ [1,3], & [2,3], & [3,3]-g, & \ldots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0.$$

This equation furnishes n values $g_1, g_2, \ldots g_n$ of g. Each root g_λ, according to (20), corresponds to a system of values,

$$(22)\qquad \alpha_{1,\lambda}, \alpha_{2,\lambda}, \ldots \alpha_{n,\lambda},$$

for whose more exact determination we also have the equation

$$(23)\qquad \alpha_{1,\lambda}^2+\alpha_{2,\lambda}^2+\alpha_{3,\lambda}^2+\ldots+\alpha_{n,\lambda}^2=1.$$

Since the determinant (21) is symmetrical and also all the quantities $[\lambda, \mu]$ are real, the roots g are real. Consequently the coefficients α of the substitution (16) are also real, and when the g's are all different, it is easy to show that they satisfy the conditions (18). But when (21) has a multiple root g_λ, apparently the transformotion becomes illusory because in this case each of the equal roots g_λ appears by reason of (20) and (23) to correspond to only one system of values (22). It has been shown that this is not so. For if all the $[\lambda, \mu]$ are real, as is here the case, not only the determinant (21) but all of its minors to the $(\lambda-1)^{\text{th}}$ degree vanish when there is a multiple root g_λ, so that, from (20) and (23), as many systems (22) appear for these roots as equals the multiplicity of g_λ. Hence, the substitution (16) is always possible, and it is indifferent whether (21) has equal roots or not.

The quantities g are also all positive, because φ represents an essentially positive quadratic quantic. They are all differ-

ent for the solar system as is shown by the substitution of numerical values dependent only on the masses and major axes for the $[\lambda, \mu]$. While we do not know the masses of all the planets with sufficient exactness, yet it has been proven that the play permitted by this imperfect knowledge is not large enough to equalize two of the quantities g. It appears as if Lagrange and Laplace dreaded this limiting case, though it would bring no complication with it, as has been shown by the later investigations concerning the transformation of a quadratic quantic, by means of an orthogonal substitution, into a sum of squares.

A more detailed accouut of the problem of this transformation and a remarkably simple way of numerically solving (21), may be found in Jacobi's article, *Ueber ein leichtes Verfahren, die in der Theorie der Säcularstörungen vorkommenden Gleichungen numerisch aufzulösen.* (*Crelle's Journal*, Bd. 30.)

We will now assume that the transformation (17) has been completed, that is, that the quatities g and α are determined and we will show that the solution of the differential equations (9) at once follows. For this purpose we will use the same orthogonal substitution (16) to transform $H_1, H_2, \ldots$ into new variables $M_1, M_2, \ldots$ and correspondingly $L_1, L_2, \ldots$ into $N_1, N_2, \ldots$ Then

$$(24) \qquad \left\{ \begin{array}{l} H_1 = \alpha_{1,1} M_1 + \alpha_{1,2} M_2 + \alpha_{1,3} M_3 + \ldots \\ H_2 = \alpha_{2,1} M_1 + \alpha_{2,2} M_2 + \alpha_{2,3} M_3 + \ldots \\ \ldots\ldots\ldots\ldots\ldots \\ L_1 = \alpha_{1,1} N_1 + \alpha_{1,2} N_2 + \alpha_{1,3} N_3 + \ldots \\ L_2 = \alpha_{2,1} N_1 + \alpha_{2,2} N_2 + \alpha_{2,3} N_3 + \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{array} \right.$$

and, conversely,

$$(25) \qquad \left\{ \begin{array}{l} M_1 = \alpha_{1,1} H_1 + \alpha_{2,1} H_2 + \alpha_{3,1} H_3 + \ldots \\ M_2 = \alpha_{1,2} H_1 + \alpha_{2,2} H_2 + \alpha_{3,2} H_3 + \ldots \\ \ldots\ldots\ldots\ldots\ldots \\ N_1 = \alpha_{1,1} L_1 + \alpha_{2,1} L_2 + \alpha_{3,1} L_3 + \ldots \\ N_2 = \alpha_{1,2} L_1 + \alpha_{2,2} L_2 + \alpha_{3,2} L_3 + \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{array} \right.$$

The linear transformation (24) of the $2n$ variables

$$H_1, H_2, \ldots \quad L_1, L_2, \ldots$$

into the $2n$ new variables

$$M_1, M_2, \ldots, \quad N_1, N_2, \ldots$$

is canonical. That is, the Poisson expressions

$$\sum_{\mu}\left(\frac{\partial H_\lambda}{\partial M_\mu}\frac{\partial L_\lambda}{\partial N_\mu}-\frac{\partial H_\lambda}{\partial N_\mu}\frac{\partial L_\lambda}{\partial M_\mu}\right)$$

equal unity, while all the other Poisson expressions vanish, as appears at once from (18). Equations (9) then become

$$\frac{dM_\lambda}{dt}=\frac{\partial W}{\partial N_\lambda}, \quad \frac{dN_\lambda}{dt}=-\frac{\partial W}{\partial M_\lambda}. \tag{26}$$

But by (15a), the function W expressed in the new variables M and N now becomes

$$W=\tfrac{1}{2}(g_1 M_1^2+g_2 M_2^2+\ldots)+\tfrac{1}{2}(g_1 N_1^2+g_2 N_2^2+\ldots)$$

The equations (26) now become

$$\frac{dM_\lambda}{dt}=g_\lambda N_\lambda, \quad \frac{dN_\lambda}{dt}=-g_\lambda M_\lambda. \tag{27}$$

The $2n$ differential equations (26) or (27) separate into n pairs which can be integrated without difficulty. It follows that

$$\left.\begin{cases} M_\lambda=K_\lambda \sin(g_\lambda t+\delta_\lambda) \\ N_\lambda=K_\lambda \cos(g_\lambda t+\delta_\lambda) \end{cases}\right\} (\lambda=1, 2, \ldots n), \tag{28}$$

in which K_λ and δ_λ are constants of integration. By the substitution of (28) in (24), we finally get the solution of the differential equations (7) or (3) in the form

$$\begin{cases} H_1=h_1\sqrt{m_1\sqrt{\mu_1 a_1}}=a_{1,1}K_1\sin(g_1 t+\delta_1)+a_{1,2}K_2\sin(g_2 t+\delta_2)+\ldots \\ H_2=h_2\sqrt{m_2\sqrt{\mu_2 a_2}}=a_{2,1}K_1\sin(g_1 t+\delta_1)+a_{2,2}K_2\sin(g_2 t+\delta_2)+\ldots \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ L_1=l_1\sqrt{m_1\sqrt{\mu_1 a_1}}=a_{1,1}K_1\cos(g_1 t+\delta_1)+a_{1,2}K_2\cos(g_2 t+\delta_2)+\ldots \\ L_2=l_2\sqrt{m_2\sqrt{\mu_2 a_2}}=a_{2,1}K_1\cos(g_1 t+\delta_1)+a_{2,2}K_2\cos(g_2 t+\delta_2)+\ldots \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{cases} \tag{29}$$

These equations represent the complete solution of the differential equations (3) and give the secular values of the eccentricities and longitudes of perihelia. We will draw some conclusions from them. If in (29) we substitute the values of h and l

$$h_\lambda = e_\lambda \sin \pi_\lambda, \quad l_\lambda = e_\lambda \cos \pi_\lambda,$$

and add the squares of the results, we get

$$(30) \quad e_1^2 m_1 \sqrt{\mu_1 a_1} + e_2^2 m_2 \sqrt{\mu_2 a_2} + \ldots = K_1^2 + K_2^2 + K_3^2 + \ldots$$

The constants in the second member are very small, as can be shown by substituting the known eccentricities in the first member; from this Laplace concluded that all eccentricities remain small. But Lagrange, on the other hand, showed that a relatively very small m_λ would permit e_λ to be large without making the term $e_\lambda^2 m_\lambda \sqrt{\mu_\lambda a_\lambda}$ relatively large.

Equations (29) also give

$$(31) \quad e_\lambda^2 m_\lambda \sqrt{\mu_\lambda a_\lambda} = \alpha_{\lambda,1}^2 K_1^2 + \alpha_{\lambda,2}^2 K_2^2 + \alpha_{\lambda,3}^2 K_3^2 + \ldots$$
$$+ 2\alpha_{\lambda,1}\alpha_{\lambda,2} K_1 K_2 \cos[(g_1 - g_2)t + \delta_1 - \delta_2] + \ldots$$

From this it follows that the maximum value which e_λ can never surpass is

$$\frac{[\alpha_{\lambda,1} K_1] + [\alpha_{\lambda,2} K_2] + \ldots}{\sqrt{m_\lambda \sqrt{\mu_\lambda a_\lambda}}}.$$

This expression is small for every value of λ, as has been shown by numerical trial, and it seems that all the eccentricities remain constantly within small limits. On the other hand it would be exceptional, if for an instant, an orbit were circular, for in this case h and l must simultaneously vanish.

For determining the longitudes of perihelia, we have from (29)

$$(32) \quad \tan \pi_\lambda = \frac{\alpha_{\lambda,1} K_1 \sin(g_1 t + \delta_1) + \alpha_{\lambda,2} K_2 \sin(g_2 t + \delta_2) + \ldots}{\alpha_{\lambda,1} K_1 \cos(g_1 t + \delta_1) + \alpha_{\lambda,2} K_2 \cos(g_2 t + \delta_2) + \ldots}.$$

It is here not easy to recognize the geometrical law of change of π_λ with the time t. But it can be shown, on the whole, that the perihelia advance; that is that they have a positive motion. By differentiation, it follows from (32), that

$$(33)\qquad e_\lambda^2 m_\lambda \sqrt{\mu_\lambda a_\lambda}\frac{d\pi_\lambda}{dt} = a_{\lambda,1}^2 K_1^2 g_1 + a_{\lambda,2}^2 K_2^2 g_2 + \dots$$
$$+ a_{\lambda,1} a_{\lambda,2} K_1 K_2 (g_1 + g_2) \cos[(g_1 - g_2)t + \delta_1 - \delta_2] + \dots$$

The constant terms in the second member are positive. The periodic terms are sometimes positive, sometimes negative, so that $d\pi_\lambda$ can be negative, but only at times, and as an interruption of the generally positive sign. This follows from the integration of (33), which gives

$$\int e_\lambda^2 m_\lambda \sqrt{\mu_\lambda a_\lambda}\, d\pi_\lambda = (a_{\lambda,1}^2 K_1^2 g_1 + a_{\lambda,2}^2 K_2^2 g_2 + \dots)t$$
$$+ a_{\lambda,1} a_{\lambda,2} K_1 K_2 \frac{g_1 + g_2}{g_1 - g_2} \sin[(g_1 - g_2)t + \delta_1 - \delta_2] + \dots$$

The term proportional to the time increases continously, while the periodic terms never surpass a definite limit. But there still appears a missing step in the argument leading to the conclusion that on the whole π_λ increases. For if two g's, say g_1 and g_2, are equal, the denominator $g_1 - g_2$ vanishes. Yet even in this case $d\pi_\lambda$ must, on the whole, increase. For the second member of (33) can also be written

$$[a_{\lambda,1} K_1 \sqrt{g_1} \sin(g_1 t + \delta_1) + a_{\lambda,2} K_2 \sqrt{g_2} \sin(g_2 t + \delta_2) + \dots]^2$$
$$+ [a_{\lambda,1} K_1 \sqrt{g_1} \cos(g_1 t + \delta_1) + a_{\lambda,2} K_2 \sqrt{g_2} \cos(g_2 t + \delta_2) + \dots]^2$$
$$+ a_{\lambda,1} a_{\lambda,2} K_1 K_2 (\sqrt{g_1} - \sqrt{g_2})^2 \cos[(g_1 - g_2)t + \delta_1 - \delta_2] + \dots$$

so that for $g_1 = g_2$ the last term entirely vanishes.

The addition of all the equations arising from (33), by exchanging λ gives

$$\sum_\lambda m_\lambda e_\lambda^2 \sqrt{\mu_\lambda a_\lambda}\frac{d\pi_\lambda}{dt} = K_1^2 g_1 + K_2^2 g_2 + K_3^2 g_3 + \dots,$$

and, since the second member is positive, it follows that all the perihelia cannot retrograde together. The numerical results show that in this thousand years all the perihelia have positive motion except Venus for which the motion is small and retrograde.

If in (32), the absolute value of any coefficient, say $[a_{\lambda,1} K_1]$ is greater than the sum of the absolute values of the others

$[a_{\lambda,2}K_2]+[a_{\lambda,3}K_3]+\ldots$, is easy to show that π_λ oscillates constantly about a mean value $g_1t+\delta_1$ or $g_1t+\delta_1+\pi$, according as $a_{\lambda,1}K_1$ is positive or negative. For it follows from (32), that

$$\tan[\pi_\lambda-(g_1t+\delta_1)]=\frac{A}{B},$$

where

$$A=a_{\lambda,2}K_2\sin[(g_2-g_1)t+(\delta_2-\delta_1)] \\ +a_{\lambda,3}K_3\sin[(g_3-g_1)t+(\delta_3-\delta_1)]+\ldots$$

and

$$B=a_{\lambda,1}K_1+a_{\lambda,2}K_2\cos[(g_2-g_1)t+(\delta_2-\delta_1)] \\ +a_{\lambda,3}K_3\cos[(g_3-g_1)t+(\delta_3-\delta_1)]+\ldots$$

The denominator, by the assumption made, can never vanish and hence $\tan[\pi_\lambda-(g_1t+\delta_1)]$ can never become infinite, so that this angle in fact oscillates about 0 or π and the magnitudes of the oscillations are always less than $\frac{1}{2}\pi$. If g_1 is the greatest root of (21), π_λ increases continuously.

When the equations (29) with the $2n$ constants of integration K and δ have been formed, the next step is to determine these constants, when the eccentricities and longitudes of perihelia, and therefore the h's and l's or the H's and L's, have been given for any instant. This is best done by equations (25), which by the substitutions (28) become

$$(34)\qquad \begin{cases} K_\lambda\sin(g_\lambda t+\delta_\lambda)=a_{1,\lambda}H_1+a_{2,\lambda}H_2+\ldots+a_{n,\lambda}H_n, \\ K_\lambda\cos(g_\lambda t+\delta_\lambda)=a_{1,\lambda}L_1+a_{2,\lambda}L_2+\ldots+a_{n,\lambda}L_n. \end{cases}$$

By division these give

$$(35)\qquad \tan(g_\lambda t+\delta_\lambda)=\frac{a_{1,\lambda}H_1+a_{2,\lambda}H_2+\ldots}{a_{1,\lambda}L_1+a_{2,\lambda}L_2+\ldots}.$$

When δ_λ has been found from (35), either of equations (34) will give the arbitrary factor K_λ. Moreover the determination of δ_λ is double, since δ_λ can be increased by π whereby the sign of K_λ will be changed.

This completes the theory of the secular variation of the eccentricities and longitudes of perihelia. There are corresponding developments for the inclinations and the longitudes of the nodes. The part of the perturbing function W which is concerned in this, is

$$-\Sigma \tfrac{1}{8} III_1(a_\lambda, a_\mu)\left[(p_\lambda - p_\mu)^2 + (q_\lambda - q_\mu)^2\right].$$

By the introduction of new variables P_λ and Q_λ, by the equations

$$(36) \qquad p_\lambda \sqrt{m_\lambda \sqrt{\mu_\lambda a_\lambda}} = P_\lambda, \quad q_\lambda \sqrt{m_\lambda \sqrt{\mu_\lambda a_\lambda}} = Q_\lambda,$$

the differential equations between p and q become

$$(37) \qquad \frac{dP_\lambda}{dt} = \frac{\partial W}{\partial Q_\lambda}, \quad \frac{dQ_\lambda}{dt} = -\frac{\partial W}{\partial P_\lambda}.$$

Instead of the quadratic function φ introduced in (15), a similar one is here used,

$$(38) \qquad \psi(x_1, x_2, \ldots x_n) = |1,1| x_1^2 + |2,2| x_2^2 + \ldots \\ + 2|1,2| x_1 x_2 + \ldots$$

where, for brevity,

$$|1,1| = -[1,1], \quad |2,2| = -[2,2] \ldots,$$

where $[1,1], [2,2], \ldots$ have the values previously given in (12), and

$$(39) \qquad |1,2| = \tfrac{1}{4} III_1(a_1, a_2) \frac{\sqrt{m_1 m_2}}{\sqrt[4]{\mu_1 a_1 \mu_2 a_2}}.$$

The transformation formulas (16) here become

$$(40) \qquad \begin{cases} x_1 = \beta_{1,1} y_1 + \beta_{1,2} y_2 + \ldots \\ x_2 = \beta_{2,1} y_1 + \beta_{2,2} y_2 + \ldots \\ \cdot \qquad \cdot \qquad \cdot \end{cases}$$

and ψ passes into

$$(41) \qquad \psi(x_1, x_2, \ldots x_n) = \gamma_1 y_1^2 + \gamma_2 y_2^2 + \gamma_3 y_3^2 + \ldots$$

The final equation (21) is, for this case

$$(42) \qquad 0 = \begin{vmatrix} |1,1| - \gamma, & |2,1|, & |3,1|, & \ldots\ldots \\ |1,2|, & |2,2| - \gamma, & |3,2|, & \ldots\ldots \\ |1,3|, & |2,3|, & |3,3| - \gamma, & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots \end{vmatrix}$$

This gives the n roots $\gamma_1, \gamma_2, \gamma_3, \ldots \gamma_n$. It can at once be shown that one of these roots $= 0$. The function ψ, like the function

$$-\Sigma \tfrac{1}{8} m_\lambda m_\mu III_1(a_\lambda, a_\mu)\left[(p_\lambda - p_\mu)^2\right],$$

is essentially negative, that is, it is negative for all systems of values of x which do not vanish. The only exception is when the p's are all equal, that is, when the x's are proportional to

$$(43) \qquad \sqrt{m_1\sqrt{\mu_1 a_1}}, \quad \sqrt{m_2\sqrt{\mu_2 a_2}}, \ldots$$

Then ψ can be written in the form

$$\psi = -\Sigma \tfrac{1}{4} III_1(a_\lambda, a_\mu) m_\lambda m_\mu \left(\frac{x_\lambda}{\sqrt{m_\lambda\sqrt{\mu_\lambda a_\lambda}}} - \frac{x_\mu}{\sqrt{m_\mu\sqrt{\mu_\mu a_\mu}}}\right)^2.$$

It follows at once from this form that the determinant of ψ vanishes. But this happens for (42) when $\gamma = 0$, so that in fact one root, say γ_n, vanishes.

From the system corresponding to (20), it follows that this root is

$$(44) \qquad \beta_{1,n} : \beta_{2,n} : \ldots = \sqrt{m_1\sqrt{\mu_1 a_1}} : \sqrt{m_2\sqrt{\mu_2 a_2}} : \ldots$$

All the other roots, $\gamma_1, \gamma_2, \ldots \gamma_{n-1}$, are negative and we have now to operate with these exactly as with the roots of equation (21).

The equations corresponding to (29) are

$$(45) \quad \begin{cases} p_1\sqrt{m_1\sqrt{\mu_1 a_1}} = \beta_{1,1}K_1' \sin(\gamma_1 t + \delta_1') \\ \qquad\qquad + \beta_{1,2}K_2' \sin(\gamma_2 t + \delta_2') + \ldots + \beta_{1,n}K_n' \sin(\delta_n'), \\ p_2\sqrt{m_2\sqrt{\mu_2 a_2}} = \beta_{2,1}K_1' \sin(\gamma_1 t + \delta_1') \\ \qquad\qquad + \beta_{2,2}K_2' \sin(\gamma_2 t + \delta_2') + \ldots + \beta_{2,n}K_n' \sin(\delta_n'), \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ q_1\sqrt{m_1\sqrt{\mu_1 a_1}} = \beta_{1,1}K_1' \cos(\gamma_1 t + \delta_1') \\ \qquad\qquad + \beta_{1,2}K_2' \cos(\gamma_2 t + \delta_2') + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

From (44) it follows that in all the p's there is one and the same constant term

$$P = \frac{\beta_{1,n}}{\sqrt{m_1\sqrt{\mu_1 a_1}}} K_n' \sin\delta_n' = \frac{\beta_{2,n}}{\sqrt{m_2\sqrt{\mu_2 a_2}}} K_n' \sin\delta_n' = \ldots,$$

and, likewise, in all the q's one and the same constant term

$$Q = \frac{\beta_{1,n}}{\sqrt{m_1\sqrt{\mu_1 a_1}}} K_n' \cos\delta_n'.$$

Since the invariable plane undergoes no secular variations, these two constants may be made to vanish by selecting it for the xy plane. By the use of (44), equations (45) give

$$P = \frac{\beta_{1,n}^{2}}{m_1\sqrt{\mu_1 a_1}}(p_1 m_1\sqrt{\mu_1 a_1} + p_2 m_2\sqrt{\mu_2 a_2} + \ldots),$$
$$Q = \frac{\beta_{1,n}^{2}}{m_1\sqrt{\mu_1 a_1}}(q_1 m_1\sqrt{\mu_1 a_1} + q_2 m_2\sqrt{\mu_2 a_2} + \ldots).$$

The sums in the parentheses represent the second members of (7) and (8), § 30, which become zero when the invariable plane is the xy plane.

Since all the roots of (42) are negative except $\gamma_n = 0$, the nodes must, on the whole, have a retrograde motion. This is the case at the present time for each of the planets.

Lagrange and Laplace gave unsymmetrical determinants, for the symmetrical ones (21) and (42), but as Jacobi has shown, they can be easily reduced to the symmetrical ones. If the elements of the last are real, as is necessary if g and γ are real, it follows that all planets must revolve about the sun in the same direction. It does not follow from this that a reversal of the motion of one or more planets would make several g's and γ's imaginary. Retrograde motion for Mercury, for example, could hardly effect this, on account of the smallness of its mass, but apparently Jupiter or Saturn could do it. Then terms of the form e^{at} would be introduced into the general integrals (3), and these terms would increase without limit. Yet it would be hasty to conclude from this, that a tendency of the eccentricities to increase continuously would finally convert the ellipses into hyperbolas. The equations (3) are only approximate and as the eccentricities increase they become less exact. In this case, it may be fairly concluded that the eccentricities would oscillate between wider limits, but it is not correct to say that the present arrangement of our system would be destroyed.

In this section h, l, p, q are sums of terms, each of which depends on the sine or cosine of an angle proportional to the time. All these terms are, therefore, periodic, and return to the same values after definite intervals, which, however, are to be reckoned in thousands of years. If g is the factor by which the time t of any term is multiplied, then the duration of the period of this term is $\frac{2\pi}{g}$. Since the quantities g are very small, the periods are very great, in fact, numerical computation shows

that the shortest of the periods embraces about 50,000 years. It appears, therefore, that the relatively short interval during which the heavens have been continuously observed, is far from sufficient to prove by observation the remarkably simple laws which have been derived.

32. The Secular Variations of the Mean Longitude.

In § 30 it was shown that the major axes undergo no secular changes. In § 31 the secular values of the eccentricities, inclinations, longitudes of perihelia, and longitudes of the nodes were given. We now proceed to the secular values of the mean longitudes by simple quadratures. The proper formulas are

$$
(1) \qquad m_\lambda \frac{d\zeta_\lambda}{dt} = m_\lambda n_\lambda - 2\sqrt{\frac{a_\lambda}{\mu_\lambda}} \frac{\partial W}{\partial a_\lambda}
+ \sqrt{\frac{1-e_\lambda^2}{a_\lambda \mu_\lambda}} \frac{(1-\sqrt{1-e_\lambda^2})}{e_\lambda} \frac{\partial W}{\partial e_\lambda}
+ \frac{1-\cos i_\lambda}{\sin i_\lambda \sqrt{\mu_\lambda a_\lambda (1-e_\lambda^2)}} \frac{\partial W}{\partial i_\lambda},
$$

in which W has the value given by (1), § 31. If we limit ourselves to the second powers of the eccentricities and inclinations, equation (1) becomes

$$
\begin{aligned}
(2) \quad \frac{d\zeta_\lambda}{dt} = n_\lambda &- \sqrt{\frac{a_\lambda}{\mu_\lambda}} \sum_\mu m_\mu \frac{\partial (I_0(a_\lambda, a_\mu))}{\partial a_\lambda} \\
&- 2\sqrt{\frac{a_\lambda}{\mu_\lambda}} \sum_\mu m_\mu \left[\tfrac{1}{8} \frac{\partial (III_1(a_\lambda, a_\mu))}{\partial a_\lambda} (h_\lambda^2 + l_\lambda^2 + h_\mu^2 + l_\mu^2) \right. \\
&\qquad \left. - \tfrac{1}{4} \frac{\partial (III_2(a_\lambda, a_\mu))}{\partial a_\lambda} (h_\lambda h_\mu + l_\lambda l_\mu) \right] \\
&+ \frac{1}{\sqrt{\mu_\lambda a_\lambda}} \sum_\mu m_\mu \left[\tfrac{1}{8} III_1(a_\lambda, a_\mu)(h_\lambda^2 + l_\lambda^2) \right. \\
&\qquad \left. - \tfrac{1}{8} III_2(a_\lambda, a_\mu)(h_\lambda h_\mu + l_\lambda l_\mu) \right] \\
&+ 2\sqrt{\frac{a_\lambda}{a_\mu}} \sum_\mu m_\mu \left[\tfrac{1}{8} \frac{\partial (III_1(a_\lambda, a_\mu))}{\partial a_\lambda} ((p_\lambda - p_\mu)^2 + (q_\lambda - q_\mu)^2) \right] \\
&+ \frac{1}{\sqrt{\mu_\lambda a_\lambda}} \sum_\mu m_\mu \left[\tfrac{1}{8} III_1(a_\lambda, a_\mu)(-p_\lambda^2 - q_\lambda^2 + p_\lambda p_\mu + q_\lambda q_\mu) \right].
\end{aligned}
$$

The first line of the second member of this equation, in which μ takes all values from 1 to n exclusive of λ, is constant. The two following terms are homogenous quadratic functions of h and l. If the values of h and l from (29), § 31, are substituted, the terms become homogenous quadratic functions of K. If they are arranged in terms of the cosines and sines of the angle $(gt+\delta)$ they give, first, constant terms; second, terms of the form $\Sigma a \cos[(g_\alpha - g_\beta)t + (\delta_\alpha - \delta_\beta)]$, where a is a constant coefficient. Constant terms and terms of the form $\Sigma b \cos[(\gamma_\alpha - \gamma_\beta)t + (\delta_\alpha' - \delta_\beta')]$ appear in the fourth and fifth lines. If all the constant terms are collected and their sum represented by c_λ, equation (2) takes the form

$$(3) \qquad \frac{d\zeta_\lambda}{dt} = n_\lambda + c_\lambda + \Sigma a \cos[(g_\alpha - g_\beta)t + (\delta_\alpha - \delta_\beta)] + \Sigma b \cos[(\gamma_\alpha - \gamma_\beta)t + (\delta_\alpha' - \delta_\beta')].$$

A very approximate value of c_λ is

$$(4) \qquad c_\lambda = -\sqrt{\frac{a_\lambda}{\mu_\lambda}} \sum_\mu m_\mu \frac{\partial (I_0(a_\lambda, a_\mu))}{\partial a_\lambda},$$

for the other terms of c_λ arising from the other terms of $\frac{d\zeta_\lambda}{dt}$ are, by what precedes, homogeneous, quadratic functions of the small quantities K and K', and are, therefore, of the second order with reference to the eccentricities and inclinations, when compared with the term of c_λ given in (4).

By integration, equation (3) gives at once

$$(5) \quad \zeta_\lambda = (n_\lambda + c_\lambda)\, t + \sum \frac{a}{g_\lambda - g_\beta} \sin[(g_\alpha - g_\beta)t + (\delta_\alpha - \delta_\beta)] + \sum \frac{b}{\gamma_\alpha - \gamma_\beta} \sin[(\gamma_\alpha - \gamma_\beta)t + (\delta_\alpha' - \delta_\beta')] + \varepsilon_\lambda,$$

where ε_λ is a constant of integration. Thus the expression for the secular mean longitude ζ_λ consists of a term proportional to the time and of periodic terms whose periods are compounded of the secular periods of the eccentricities and inclinations. The coefficients $\frac{a}{g_\alpha - g_\beta}$ and $\frac{b}{\gamma_\alpha - \gamma_\beta}$ do not contain the disturbing masses as factors, for while these occur in a and b, g and γ are

also proportional to them. On the other hand, they are of the second order with reference to K and K' and, as numerical computation shows, they are so small that they may be neglected.

While these terms may be neglected in the planetary theories, they play an important part in the closely related lunar theory. They are here of considerable size and from them Laplace obtained an explanation of the secular acceleration of the moon's mean motion, discovered by observation. The numerical values given by him, in this case, are doubted by later astronomers.

Omitting the secular-periodic terms, we have

$$\zeta_\lambda = (n_\lambda + c_\lambda)t + \varepsilon_\lambda, \tag{6}$$

that is, exactly the result of (2), § 22, except that $n_\lambda = \sqrt{\dfrac{M+m_\lambda}{a_\lambda^3}}$ must be increased by a small term which, by (4), is proportional to the disturbing masses. Kepler's third law, already modified by using $\mu_\lambda = M + m_\lambda$, must have a further correction on account of the small term c_λ, if the major axis is to be obtained from the mean periodic time or *vice versa*. The periodic time is

$$T_\lambda = \frac{2\pi}{n_\lambda + c_\lambda} = \frac{2\pi}{\sqrt{\dfrac{M+m_\lambda}{a_\lambda^3}} + c_\lambda}. \tag{7}$$

And, finally, an important conclusion from this section is that

The year of each planet, obtained as the mean of a great number of observations, is constant.

33. The Periodic Terms in the Elements. A Combination of the Theory of Absolute Perturbations with the Theory of Variation of Elements.

In the preceding sections the secular values of the elements $a, e, \pi, i, \Omega, \zeta$ have been obtained by limiting the perturbing function to terms of the second degree in the eccentricities and inclinations. If the secular values were exact they would differ from the actual elements by the small periodic quantities already given. If the secular elements are designated by $[a], [e], \ldots$ and the periodic terms by $(a), (e), \ldots,$ the true elements will be

$$(1)\qquad \begin{cases} a = [a] + (a), \\ e = [e] + (e), \\ \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\ \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \end{cases}$$

The periodic terms arise from the periodic terms of the perturbing function when the secular values $[a], [e], \ldots [\zeta]$ are substituted for the elements $a, e, \ldots \zeta$. Let

$$(2)\qquad k\,{\cos \atop \sin}(\mathfrak{a}_1[\zeta_1] + \mathfrak{a}_2[\zeta_2])$$

be such a periodic term of R_1. If, for brevity, we put

$$(3)\qquad \begin{cases} \mathfrak{a}_1[\zeta_1] + \mathfrak{a}_2[\zeta_2] = \lambda, \\ \mathfrak{a}_1[n_1] + \mathfrak{a}_2[n_2] = \alpha, \end{cases}$$

then by (8), § 28, the corresponding term

$$(4)\qquad \begin{cases}
\text{in } (a_1) \\
\quad = 2\sqrt{\dfrac{[a_1]}{\mu_1}}\, k\, \dfrac{\mathfrak{a}_1}{\alpha} {\cos \atop \sin}(\lambda), \\
\text{in } (e_1) \\
\quad = \dfrac{-\sqrt{1-[e_1]^2}\,(1-\sqrt{1-[e_1]^2})}{[e_1]\sqrt{[a_1]\mu_1}}\, k\, \dfrac{\mathfrak{a}_1}{\alpha} {\cos \atop \sin}(\lambda) \\
\quad\quad - \dfrac{1}{[e_1]}\sqrt{\dfrac{1-[e_1]^2}{[a_1]\mu_1}}\, \dfrac{\partial k}{\partial[\pi_1]}\, \dfrac{1}{\alpha} {\sin \atop -\cos}(\lambda), \\
\text{in } (\pi_1) \\
\quad = \dfrac{1}{[e_1]}\sqrt{\dfrac{1-[e_1]^2}{[a_1]\mu_1}}\, \dfrac{\partial k}{\partial[e_1]}\, \dfrac{1}{\alpha} {\sin \atop -\cos}(\lambda) \\
\quad\quad + \dfrac{1-\cos[i_1]}{\sin[i_1]\sqrt{[a_1]\mu_1(1-[e_1]^2)}}\, \dfrac{1}{\alpha}\, \dfrac{\partial k}{\partial[i_1]} {\sin \atop -\cos}(\lambda), \\
\text{in } (i_1) \\
\quad = -\dfrac{1-\cos[i_1]}{\sin[i_1]\sqrt{\mu_1[a_1](1-[e_1]^2)}}\, k\, \dfrac{\mathfrak{a}_1}{\alpha} {\cos \atop \sin}(\lambda) \\
\quad\quad - \dfrac{(1-\cos[i_1])\dfrac{\partial k}{\partial[\pi_1]} + \dfrac{\partial k}{\partial[\Omega_1]}}{\sqrt{\mu_1[a_1](1-[e_1]^2)}\sin[i_1]}\, \dfrac{1}{\alpha} {\sin \atop -\cos}(\lambda), \\
\text{in } (\Omega_1) \\
\quad = \dfrac{\dfrac{\partial k}{\partial[i_1]}}{\sin[i_1]\sqrt{\mu_1[a_1](1-[e_1]^2)}}\, \dfrac{1}{\alpha} {\sin \atop -\cos}(\lambda).
\end{cases}$$

The corresponding periodic term of ζ_1 remains to be developed. The first term of $\frac{d\zeta_1}{dt}$ in (2), § 32, namely n_1, does not contain the disturbing masses as factors, and in the integral

$$\int n_1\, dt$$

we must take account of the periodic terms (n_1). Now

$$n_1 = \sqrt{\frac{\mu_1}{a_1^{\,3}}} = \sqrt{\frac{\mu_1}{([a_1]+(a_1))^3}} = [n_1]\left(1 - \frac{3\,(a_1)}{2\,[a_1]}\right),$$

and, therefore,

$$\int n_1\, dt = [n_1]t - \frac{3[n_1]}{2[a_1]}\int (a_1)\, dt.$$

The term of the integral corresponding to the term (2) in the perturbing function is, by the first of equations (4),

$$= 2\sqrt{\frac{[a_1]}{\mu_1}}\, k\, \frac{a_1}{a^2}\,\frac{\sin}{-\cos}(\lambda),$$

and hence the periodic term in ζ corresponding to (2) is

$$\begin{aligned}
(5) \qquad = &-\frac{3}{[a_1]^2}\, k\, \frac{a_1}{a^2}\,\frac{\sin}{-\cos}(\lambda) - 2\sqrt{\frac{[a_1]}{\mu_1}}\,\frac{\partial k}{[\partial a_1]}\,\frac{1}{a}\,\frac{\sin}{-\cos}(\lambda)\\
&+\frac{\sqrt{1-[e_1]^2}\,(1-\sqrt{1-[e_1]^2})}{[e_1]\sqrt{[a_1]\,\mu_1}}\,\frac{\partial k}{\partial [e_1]}\,\frac{1}{a}\,\frac{\sin}{-\cos}(\lambda)\\
&+\frac{1-\cos[i_1]}{\sin[i_1]\sqrt{\mu_1[a_1](1-[e]^2)}}\,\frac{\partial k}{\partial [i_1]}\,\frac{1}{a}\,\frac{\sin}{-\cos}(\lambda).
\end{aligned}$$

By the substitution of these values in (1), we get the elements as completely determined functions of the time and these can, therefore, be computed for any given instant. From the elements the coordinates can be obtained by § 5, exactly as if the elements were constant.

The elements are thus divided into two parts, the secular and the periodic. The last are constantly proportional to the disturbing masses and are therefore small. According to the developments in §§ 30 and 31, the first are also periodic, but their periods are quite different, since the mean longitudes appear in the trigonometrical functions in the periodic terms. The secular periods embrace thousands of years and cause a very slow

change in the secular values of the elements, except, of course, the terms in $[\zeta]$ which are proportional to the time. We will now see how the periodic terms may be very conveniently separated from the secular in the final expressions for the coordinates.

Let x be any coordinate. It is a function of the elements, therefore,

$$x = x(a, e, i, \pi, \Omega, \zeta),$$

or, by substituting the values of the elements from (1),

$$(6) \qquad x = x([a] + (a), [e] + (e), \ldots, [\zeta] + (\zeta)).$$

If the second member of this equation be developed in powers of the quantities (a), (e), ..., which are small and proportional to the disturbing masses, and limit the development to their first powers, we get

$$(7) \qquad x = x([a], [e], [i], [\pi], [\Omega], [\zeta]) + \frac{\partial x}{\partial[a]}(a) + \frac{\partial x}{\partial[e]}(e) + \ldots + \frac{\partial x}{\partial[\zeta]}(\zeta).$$

The terms $\frac{\partial x}{\partial[a]}(a) + \ldots$ are, by § 22 and §28, exactly the same as the purely periodic terms in the absolute perturbations of the first degree. Hence (7) becomes

$$(8) \qquad x = x([a], [e], \ldots) + (\delta x).$$

From formula (8) flows the following remarkable combination of the two different methods for obtaining the motions of the planets, the theory of variation of constants and that of absolute perturbations.

Imagine first a fictitious (say a secular) planet, which moves in an ellipse with the secular mean motion of § 32, and whose elements are the secular values of the real elements. The real planet will always be a small distance from the fictitious planet the distance depending on the periodic terms. The departures of the real planet from the fictitious place may be regarded as due to the departures of the elements from their secular values, and these may be treated, according to (8), directly as absolute perturbations. In this way the variation of constants is used to

get the effects of the secular terms of the perturbing function, while the theory of absolute perturbations is applied only to its periodic terms.

It is remakable that this combination of the two methods was used earlier than that of the pure variation of the elements. Euler and Laplace recognized the fact that the secular terms of the perturbing function exercised a vital influence on this variation and they fixed their attention on this alone. Lagrange first gave the theory of the variation of elements in all its purity.

Astronomers, in calculating the positions of the planets, usually employ a combination of the two methods, and they thus get a system of formulas which are very convenient.

34. The Stability of the Solar System.

The important results, which the theory of the variation of the elements brings to light, are not only a close approximation to the truth, but they are also of the greatest importance in solving a problem of the highest interest—the problem of the stability of the solar system. With the limitation that an approximation, although next to mathematically exact, can never take the place of a mathematical certainty, the theory enables us to affirm the stability of the solar system.

The planetary system certainly undergoes important changes in the couse of centuries, yet these changes do not affect the two elements which with entire propriety have been called the principal ones,—the mean distance and the periodic time. If the last has been obtained from hundreds or thousands of observed periods, we may be positive that this mean exactly represents the invariable periodic time. This element, with Kepler's third law, corrected as in § 32, gives the exact mean distance, which is incapable of direct observation. While the directions of the axes, the eccentricities, nodes, and inclinations may undergo great changes in the course of time, yet the stability of the system is even here assured in so far as it depends on the eccentricities which always remain small, and upon the inclinations which are small when referred to the invariable plane.

The recognition of the stability of the system is the finest result reached by the investigations of Lagrange and Laplace. The dread that the mutual attraction might eventually cause a collision is completely relieved. The planets will always circulate as regularly about the sun as if each alone were the only planetary member of the system.

35. The Effect of Neglected Secular Terms of the Perturbing Function whose Degrees with Respect to the Eccentricities and Inclinations are Higher than the Second.

In § 31 the secular values of the elements were developed for the terms of lower degree of the perturbing function. By the omission of terms of higher degree it was possible to change the differential equations between h, l, p, q, and the time t into linear equations with constant coefficients. If the eccentricities and inclinations are very small, the neglected terms are also very small, as compared with those not neglected. At the same time, it is evidently possible that a very small additional term in a differential equation may have a notable effect in the integral, especially if the time be sufficiently extended.

So far as the author knows, Leverrier was the first to take into account the terms of higher order in the secular value of the perturbing function. This was done in his work,—*Intégration des équations différentielles, dont dépendent les inégalités séculaires, en tenant compte des termes, qui sont du troisième ordre par rapport aux excentricités et aux inclinaisons.*

If W is developed to the fourth degree inclusive, then

$$W = W_0 + W_2 + W_4, \tag{1}$$

where W_0, W_2, W_4 are the terms of the 0^{th}, 2^{nd} and 4^{th} orders. If h, l, p, q are introduced, (2), § 31, gives at once W_0 and W_2. The formation of W_4 is more complicated. In every case h, l, p, q occur only in the combinations

$$\begin{gathered} h_\lambda^2 + l_\lambda^2, \quad h_\lambda h_\mu + l_\lambda l_\mu, \quad p_\lambda^2 + q_\lambda^2, \quad p_\lambda p_\mu + q_\lambda q_\mu, \\ h_\lambda p_\lambda + l_\lambda q_\lambda, \quad h_\lambda p_\mu + l_\lambda q_\mu, \end{gathered} \tag{2}$$

and in such a way that each term is even both in reference to h and l, and also p and q. W_4 can then be represented as a homogeneous quadratic function of the combinations (2), but never so that one of the first four is multiplied by one of the last two.

When the second members of equations (3), § 31, include the terms of the third degree, they become

$$(3)\quad \begin{cases} m\sqrt{\mu a}\,\dfrac{dh}{dt} = \dfrac{\partial W_2}{\partial l} + \Big[-\dfrac{h^2+l^2}{2}\dfrac{\partial W_2}{\partial l} \\ \qquad\qquad + \dfrac{l}{2}\Big(p\dfrac{\partial W_2}{\partial p} + q\dfrac{\partial W_2}{\partial q}\Big) + \dfrac{\partial W_4}{\partial l}\Big], \\ m\sqrt{\mu a}\,\dfrac{dl}{dt} = -\dfrac{\partial W_2}{\partial h} + \Big[\dfrac{h^2+l^2}{2}\dfrac{\partial W_2}{\partial h} \\ \qquad\qquad - \dfrac{h}{2}\Big(p\dfrac{\partial W_2}{\partial p} + q\dfrac{\partial W_2}{\partial q}\Big) - \dfrac{\partial W_4}{\partial h}\Big], \\ m\sqrt{\mu a}\,\dfrac{dp}{dt} = \dfrac{\partial W_2}{\partial q} + \Big[\dfrac{h^2+l^2-p^2-q^2}{2}\dfrac{\partial W_2}{\partial q} \\ \qquad\qquad - \dfrac{p}{2}\Big(l\dfrac{\partial W_2}{\partial h} - h\dfrac{\partial W_2}{\partial l}\Big) + \dfrac{\partial W_4}{\partial q}\Big], \\ m\sqrt{\mu a}\,\dfrac{dq}{dt} = -\dfrac{\partial W_2}{\partial p} + \Big[\dfrac{p^2+q^2-h^2-l^2}{2}\dfrac{\partial W_2}{\partial p} \\ \qquad\qquad - \dfrac{q}{2}\Big(l\dfrac{\partial W_2}{\partial h} - h\dfrac{\partial W_2}{\partial l}\Big) - \dfrac{\partial W_4}{\partial p}\Big]. \end{cases}$$

These equations are to be formed for each planet, and the problem is to integrate them, even if only approximately. If the terms in brackets were equal to 0, then h, l, p, q would have the forms given in § 31. We will use Lagrange's method and assume that the same form occurs here with the difference that K, δ, K', δ' are no longer constants but undetermined functions of t. (The α's and g's and the β's and λ's in the equations (29) and (45), § 31 remain invariable since they are not constants of integration but depend upon the invariable axes.)

If we put

$$(4)\qquad G_\lambda = g_\lambda t + \delta_\lambda, \quad \Gamma_\lambda = \gamma_\lambda t + \delta_\lambda',$$

we have, by § 31,

$$(5)\qquad W_2 = \tfrac{1}{2}(g_1 K_1^2 + g_2 K_2^2 + g_3 K_3^2 + \dots) + \tfrac{1}{2}(\gamma_1 K_1'^2 + \gamma_2 K_2'^2 + \gamma_3 K_3'^2 + \dots).$$

W_4 takes the form

(6) $$W_4 = k_0 + \Sigma k \cos \lambda,$$

where the coefficients k are homogeneous functions of the fourth degree and λ an angle of the form

(7) $$\lambda = a_1 G_1 + a_2 G_2 + \ldots + b_1 \Gamma_1 + b_2 \Gamma_2 + \ldots$$

The quantities a and b are here positive or negative whole numbers, 0 not included, which satisfy the conditions

(8) $$a_1 + a_2 + \ldots + b_1 + b_2 + \ldots = 0,$$

(9) $$a_1 + a_2 + \ldots + a_n = \text{an even number},$$

(10) $$[a_1] + [a_2] + \ldots + [b_1] + [b_2] + \ldots = 4, \text{ or } 2, \text{ or } 0.$$

If for brevity the bracketed quantities in (3) be represented by

(11) $$[H], [L], [P], [Q],$$

we obtain by the substitution of (29) and (45), § 31,

(12) $$[H] = \Sigma k \cos \lambda,$$

where the k's are homogeneous functions of the 3rd degree in K and the λ's angles of the form (7), except that the second member of (8) equals 1, the sum (9) is odd and the sum (10) equals 3 or 1.

It likewise follows that

(13) $$[L] = -\Sigma k \sin \lambda,$$

where the k's and λ's have the same values as (12).

Finally

$$P = \Sigma k' \cos \lambda', \quad Q = -\Sigma k' \sin \lambda',$$

where the k's denote homogeneous functions of K of the third degree and the angles λ' have the form (7) except that the second member of (8) equals 1, the equation (9) remains unchanged, while the second member of (10) equals 3 or 1.

The equations (34), § 31, now become

(14) $$K_\lambda \sin G_\lambda = \sum_{\mu=1}^{\mu=n} a_{\mu,\lambda} \sqrt{m_\mu \sqrt{\mu_\mu a_\mu}}\, h_\mu,$$

(15) $$K_\lambda \cos G_\lambda = \sum_{\mu=1}^{\mu=n} a_{\mu,\lambda} \sqrt{m_\mu \sqrt{\mu_\mu a_\mu}}\, l_\mu.$$

Differentiating these equations and remembering that K and δ are variable, it follows that

$$\frac{dK_\lambda}{dt}\sin G_\lambda + K_\lambda \cos G_\lambda \frac{d\delta_\lambda}{dt} + g_\lambda K_\lambda \cos G_\lambda = \sum_{\mu=1}^{\mu=n} a_{\mu,\lambda} \sqrt{m_\mu \sqrt{\mu_\mu a_\mu}}\, \frac{dh_\mu}{dt}.$$

Substituting the value of $\frac{dh_\mu}{dt}$ from (3) and remembering that the term $g_\lambda K_\lambda \cos G_\lambda$, by § 31, must vanish because of the term $\sum_\mu \frac{a_{\mu,\lambda}}{\sqrt{m_\mu \sqrt{\mu_\mu a_\mu}}} \frac{\partial W_2}{\partial l_\mu}$, the preceding equation becomes

$$\frac{dK_\lambda}{dt}\sin G_\lambda + K_\lambda \cos G_\lambda \frac{d\delta_\lambda}{dt} = \sum_{\mu=1}^{\mu=n} \frac{a_{\mu,\lambda}}{\sqrt{m_\mu \sqrt{\mu_\mu a_\mu}}} [H_\mu],$$

and similarly,

$$\frac{dK_\lambda}{dt}\cos G_\lambda - K_\lambda \sin G_\lambda \frac{d\delta_\lambda}{dt} = \sum_{\mu=1}^{\mu=n} \frac{a_{\mu,\lambda}}{\sqrt{m_\mu \sqrt{\mu_\mu a_\mu}}} [L_\mu],$$

and, therefore,

$$(16)\quad \begin{cases} \dfrac{dK_\lambda}{dt} = \displaystyle\sum_{\mu=1}^{\mu=n} \frac{a_{\mu,\lambda}}{\sqrt{m_\mu \sqrt{\mu_\mu a_\mu}}} ([H_\mu]\sin G_\lambda + [L_\mu]\cos G_\lambda), \\ \text{and} \\ K_\lambda \dfrac{d\delta_\lambda}{dt} = \displaystyle\sum_{\mu=1}^{\mu=n} \frac{a_{\mu,\lambda}}{\sqrt{m_\mu \sqrt{\mu_\mu a_\mu}}} ([H_\mu]\cos G_\lambda - [L_\mu]\sin G_\lambda). \end{cases}$$

If the values of $[H]$ and $[L]$ are substituted from (12) and (13), the second members of these equations take the forms

$$\Sigma k \sin\lambda, \quad \Sigma k \cos\lambda,$$

where again the k's are homogeneous functions of the K's and K''s of the third degree and indeed of an even degree with reference to the K''s, and further the angle λ has the form (7). The corresponding equations are

$$(17)\begin{cases} \dfrac{dK_\lambda'}{dt} = \displaystyle\sum_{\mu=1}^{\mu=n} \frac{\beta_{\mu,\lambda}}{\sqrt{m_\mu \sqrt{\mu_\mu a_\mu}}}([P_\mu]\sin\Gamma_\lambda + [Q_\mu]\cos\Gamma_\lambda), \\ K'_\lambda \dfrac{d\delta'}{dt} = \displaystyle\sum_{\mu=1}^{\mu=n} \frac{\beta_{\mu,\lambda}}{\sqrt{m_\mu \sqrt{\mu_\mu a_\mu}}}([P_\mu]\cos\Gamma_\lambda - [Q_\mu]\sin\Gamma_\lambda), \end{cases}$$

where the second members again have the forms

$$(18) \qquad \Sigma k \sin\lambda, \quad \Sigma k \cos\lambda.$$

In order to integrate these equations approximately, distinguish the so-called secular-secular in the second member, that is, the terms independent of G and Γ from the secular-periodic terms, that is, the terms containing G and Γ. Since $\frac{dK}{dt}$ and $\frac{dK'}{dt}$ here have the form $\Sigma k \sin\lambda$, and the secular terms fail, it follows, when limited to these, that

$$\frac{dK}{dt} = 0, \quad \frac{dK'}{dt} = 0.$$

Therefore, *the former constants of integration K and K' remain constant as before.*

It is otherwise with the derivatives of the δ's and δ''s. Taking the term of (18) for which $\lambda = 0$ and representing its coefficients by k_0 and k_0', it follows that

$$K_\lambda \frac{d\delta_\lambda}{dt} = k_0,$$

$$K_\lambda \frac{d\delta_\lambda'}{dt} = k_0'.$$

The quantities k_0 and k_0' are homogenous and entire functions of the third degree in K and K', therefore constant by what precedes.

From this follows

$$\delta_\lambda = \frac{k_0}{K_\lambda} t + \varepsilon_\lambda,$$

$$\delta_\lambda' = \frac{k_0'}{K_\lambda} t + \varepsilon_\lambda'.$$

That is, *the previous constants δ_λ and δ_λ' become linear functions of the time.*

If we put

$$\frac{k_0}{K_\lambda} = \delta g_\lambda, \quad \frac{k_0'}{K_\lambda} = \delta\gamma_\lambda, \tag{19}$$

the angles G_λ and Γ_λ take the form

$$G_\lambda = (g_\lambda + \delta g_\lambda)t + \varepsilon_\lambda = g_\lambda' t + \varepsilon_\lambda, \tag{20}$$

$$\Gamma_\lambda = (\gamma_\lambda + \delta\gamma_\lambda)t + \varepsilon_\lambda' = \gamma_\lambda' t + \varepsilon_\lambda'. \tag{21}$$

The angles G_λ and Γ_λ remain linear functions of the time, but the coefficients of the time t are no longer the g's and γ's determined by (21) and (42), § 31, but they have slight additions δg and $\delta\gamma$ which are of the second order with reference to the eccentricities and inclinations.

To take account of the secular periodic terms, the principles of § 29 will be followed. In any such term as $k \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \lambda$, the secular values are to be substituted for K, K', G, Γ, and the term is then to be integrated. In this way terms arise in K and K' of the form

$$-\frac{k\cos\lambda}{a_1 g_1' + a_2 g_2' + \ldots + b_1\gamma_1' + b_2\gamma_2' + \ldots}, \tag{22}$$

while the corresponding terms in δ and δ' run

$$+\frac{k}{K}\,\frac{\sin\lambda}{a_1 g_1' + a_2 g_2' + \ldots + b_1\gamma_1' + b_2\gamma_2' + \ldots}. \tag{23}$$

We see from this that the action of the secular terms of higher order appears in two things:

First—The period of the secular angles G and Γ are changed by quantities proportional to the squares of the eccentricities and inclinations.

Second—To the original secular terms in h, l, p, q terms are added whose arguments are linear combinations of the original arguments with integral coefficients.

And here is a circumstance which at once carries certain secular-periodic into secular-secular terms. We have seen that one γ, for example γ_n, is zero. Hence $\Gamma_n = \delta_n' =$ a constant. The terms of (18) for which $\lambda = 2\gamma_n$, or $4\gamma_n$ are, therefore, to be

regarded as secular-secular terms. Hence the equations $\frac{dK}{dt}=0$ and $\frac{dK'}{dt}=0$ are not fulfilled and this is entirely correct. In fact, these two equations are not generally correct; the invariable plane should be selected as the plane of xy, because, as in § 31, the terms dependent on γ_n then vanish in the first approximation. With this selection of coordinates it is also evident that no secular terms would appear in $\frac{d\delta_n'}{dt}$, so that the new $\gamma_n' = \gamma_n + \delta\gamma_n'$ would vanish.

In the same way still higher terms could be brought into the account and, finally, a formal development of h, l, p, q would be obtained like the following. The h's and l's would take the forms

$$h = \Sigma k \sin\lambda, \quad l = \Sigma k \cos\lambda, \tag{24}$$

where λ is an angle of the form (7) with the condition that

$$a_1 + a_2 + \ldots + b_1 + b_2 + \ldots = 1,$$

and the numbers a and b including all possible integers which satisfy the condition that $a_1 + a_2 + \ldots + a_n$ is odd. By the selection of the invariable plane as that of xy, b_n is discarded. Likewise, the values of p and q have the same forms

$$p = \Sigma k' \sin\lambda', \quad q = \Sigma k' \cos\lambda', \tag{25}$$

where λ' has the same form as λ except that the sum $a_1 + a_2 + \ldots + a_n$ is even. The angles λ and λ' are linear functions of the time because G and Γ are such. The solutions (24) and (25) are, to be sure, only formal. There appear for instance, in (22) and (23), denominators which eventually vanish, or at least become very small. Leverrier and Lehmann have specified several such denominators and in combination with numerical computation have come to the conclusion that the influence of terms of higher order is decidedly greater than was supposed by Lagrange and Laplace.

If, overlooking these difficulties, the periodic terms introduced in § 33 are added to the elements, we get, when these are

introduced into the expressions for the coordinates, the following general schematic representation of the coordinates:

(26) $$x = \Sigma K \cos L, \quad y = \Sigma K \sin L,$$

(27) $$z = \Sigma K' \sin L'.$$

The K's and K''s are coefficients independent of the time and the angles L are of the following form:

(28) $$L = \mathfrak{a}_1 \zeta_1 + \mathfrak{a}_2 \zeta_2 + \ldots + \mathfrak{b}_1 G_1 + \mathfrak{b}_2 G_2 + \ldots + \mathfrak{c}_1 \Gamma_1 + \mathfrak{c}_2 \Gamma_2 + \ldots$$

The integral numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ satisfy the condition

$$\mathfrak{a}_1 + \mathfrak{a}_2 + \ldots + \mathfrak{b}_1 + \mathfrak{b}_2 + \ldots + \mathfrak{c}_1 + \mathfrak{c}_2 + \ldots = 1.$$

The angles L' are of the same form (28), except that for them the sum of the integral numbers $= 0$.

In spite of their very different origin ζ, G, and Γ have the common property that they increase proportionally to the time. We will resume this representation of the coordinates in § 40 and the following sections.

36. Terms of Long Period and the Commensurability of the Periodic Times.

In the separation of the actions of the secular and periodic terms on the elements or coordinates, a fundamental difficulty appears which the efforts of mathematicians and astronomers have not succeeded in overcoming. If

(1) $$k \begin{matrix}\cos\\ \sin\end{matrix} (\mathfrak{a}_1 \zeta_1 + \mathfrak{a}_2 \zeta_2)$$

is a periodic term of the perturbing function, then, by formulas (4), § 33, similar terms must appear in the elements. In these

(2) $$\mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2$$

appears in the denominator. In the mean longitudes the term

(3) $$-\frac{3}{a_1^2} \frac{\mathfrak{a}_1 k}{(\mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2)^2} \begin{matrix}\sin\\ -\cos\end{matrix} (\mathfrak{a}_1 \zeta_1 + \mathfrak{a}_2 \zeta_2)$$

occurs and in its denominator the square of this binomial appears.

It is these denominators that cause the difficulty mentioned. Even when n_1 and n_2 are irrational with respect to each other, a

series of whole numbers can be so determined that the equation

$$(4)\qquad \mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2 = 0$$

is almost exactly fulfilled. This is best done by developing $\frac{n_1}{n_2}$ in a continued fraction and determining approximately the numerators and denominators. The periodic perturbations, which depend on the term (1), then become disproportionately enlarged by these denominators, and although the original coefficient k may have been very small, the corresponding coefficients in these perturbations become important and may become infinite.

This difficulty is generally met as follows: The mean daily motion n in our system has the property that equation (4) is approximately or exactly fulfilled only when $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are very large numbers. In this case the coefficient k of the perturbing function, which is of the degree $[\mathfrak{a}_1 + \mathfrak{a}_2]$ with reference to the eccentricities and inclinations, becomes so extremely small that the term has no sensible influence on the elements for very long periods of time.

We will, however, assume that $\mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2$ is very small for integral numbers $\mathfrak{a}_1$ and $\mathfrak{a}_2$ which are not too large, and investigate more closely the action of the corresponding term (1) of the perturbing function on the perturbations. The period of this term is

$$(5)\qquad T = \frac{2\pi}{\mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2}.$$

First it is to be noted that such a term appears in R_1 and also in R_2, and we will compare the coefficients k in the two cases. Leaving the other planets out of consideration, we have

$$(6)\begin{cases} R_1 = m_2\left(\dfrac{1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}} - \dfrac{x_1x_2+y_1y_2+z_1z_2}{r_2^{\,3}}\right), \\ R_2 = m_1\left(\dfrac{1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}} - \dfrac{x_1x_2+y_1y_2+z_1z_2}{r_1^{\,3}}\right). \end{cases}$$

For brevity put $\mathfrak{a}_1 \zeta_1 + \mathfrak{a}_2 \zeta_2 = \lambda$. Then let the periodic term with the argument λ

$$\text{in } \frac{1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}$$
$$= P\cos\lambda + Q\sin\lambda,$$
$$\text{in } x_1x_2+y_1y_2+z_1z_2$$
$$= p\cos\lambda + q\sin\lambda.$$

P and Q, with respect to the eccentricities and inclinations, are of the degree $[\mathfrak{a}_1+\mathfrak{a}_2]$, while p and q are of the degree $[\mathfrak{a}_1]+[\mathfrak{a}_2]-2$, so that, if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are large numbers with opposite signs, (which must be the case since $\mathfrak{a}_1 n_1+\mathfrak{a}_2 n_2$), is to be very small the coefficients p and q are in general smaller than P and Q. By (16), § 26, the corresponding terms become

in R_1

$$= m_2\left[\left(P-\frac{\mathfrak{a}_2^2 n_2^2}{\mu_2}p\right)\cos\lambda+\left(Q-\frac{\mathfrak{a}_2^2 n_2^2}{\mu_2}q\right)\sin\lambda\right],$$

in R_2

$$= m_1\left[\left(P-\frac{\mathfrak{a}_1^2 n_1^2}{\mu_1}p\right)\cos\lambda+\left(Q-\frac{\mathfrak{a}_1^2 n_1^2}{\mu_1}q\right)\sin\lambda\right].$$

We can here put $\mu_1=\mu_2=M$. The factors of m_2 and m_1 then become nearly equal, for their difference is

$$= \frac{(\mathfrak{a}_1 n_1-\mathfrak{a}_2 n_2)(\mathfrak{a}_1 n_1+\mathfrak{a}_2 n_2)}{M}(p\cos\lambda+q\sin\lambda).$$

Not only are p and q small in relation to P and Q but $\mathfrak{a}_1 n_1+\mathfrak{a}_2 n_2$ must also be relatively small. Consequently, the above terms become

in R_1

$$= m_2(P'\cos\lambda+Q'\sin\lambda)=m_2 C\cos(\lambda+\varepsilon), \tag{7}$$

in R_2

$$= m_1(P'\cos\lambda+Q'\sin\lambda)=m_1 C\cos(\lambda+\varepsilon). \tag{8}$$

The terms in the mean longitudes ζ_1 and ζ_2 corresponding to the periodic perturbation terms (3), are, therefore,

$$-\frac{3}{\mathfrak{a}_1^2}\frac{\mathfrak{a}_1 m_2}{(\mathfrak{a}_1 n_1+\mathfrak{a}_2 n_2)^2}C\sin(\lambda+\varepsilon), \text{ and} \tag{9}$$

$$-\frac{3}{\mathfrak{a}_2^2}\frac{\mathfrak{a}_2 m_1}{(\mathfrak{a}_1 n_1+\mathfrak{a}_2 n_2)^2}C\sin(\lambda+\varepsilon). \tag{10}$$

Their

$$=\frac{\mathfrak{a}_1 m_2 a_2^2}{\mathfrak{a}_2 m_1 a_1^2}.$$

Since $\mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2$ is to be very small, we have, approximately

$$\frac{\mathfrak{a}_1}{\mathfrak{a}_2}=\frac{-n_2}{n_1}=-\frac{\sqrt{a_1^3}}{\sqrt{a_2^3}},$$

and the above ratio becomes

$$-\frac{m_2\sqrt{a_2}}{m_1\sqrt{a_1}}.$$

It is therefore constant and negative. If one mean longitude is increased by the periodic perturbation depending on the argument λ, the other is decreased. These deviations are inversely as the masses and the square roots of the mean distances.

As the duration of the period is long, it can only be recognized as a period, by observation, after a long series of revolutions about the sun. One revolution would appear accelerated, another retarded, until finally the relation would be reversed and the acceleration would pass into a retardation and *vice versa.*

This phenomenon is known with accuracy for Jupiter and Saturn. Representing Jupiter by the subscript 1 and Saturn by 2, we have in this case

$$5n_2-2n_1=$$

a very small fraction of n_1 and n_2 and about $=\frac{n_2}{30}$. The angle

$$\lambda=5\zeta_2-2\zeta_1$$

increases very slowly, and since Saturn's period is about thirty years, it takes about 900 years for λ to increase to 2π, and, consequently, for the restoration of the previous relation between the corresponding terms of the perturbations of the mean longitude. Since $[\mathfrak{a}_1+\mathfrak{a}_2]=3$, C in (7) and (8) is of the third degree with respect to the eccentricities and inclinations, and must be relatively small. On the other hand, the coefficients in (9) and (10) increase so much, through the square of the small denominator $5n_2-2n_1$, that they are not only appreciable, but in fact the greatest of all the periodic perturbations of the

solar system. The irregularities of the periodic times caused by these perturbations were noticed at an early time by a comparison of the periods of revolution for different epochs. It was thought that the smallness of the coefficients warranted the omission of these terms of the perturbations, and early astronomers, including such men as Euler and Lagrange, were at a loss to account for these irregularities. They suspected the existence of something foreign to Newton's law of gravitation until Laplace succeeded in showing the real cause in the appearance of the square of a small denominator, the smallness of which was due to the approximate commensurability of the two periodic times.

We will now go further and study a case, not actually planetary, but of the secondary systems of the Earth and Jupiter, where, under changed circumstances, it plays an important part.

Assume two periodic times so nearly commensurable that the coefficients of the terms (9) and (10) in the mean longitudes assume so great a magnitude that it puts in jeopardy the process of integration which depends on the smallness of the periodic perturbations and their relatively short periods. Preserving the previous symbols, and limiting the consideration to a seventh or eighth term of the perturbing function in (8), § 28, we have

$$\frac{dn_1}{dt} = -\frac{3}{2}\frac{n_1}{a_1}\frac{da_1}{dt} = \frac{3\mathfrak{a}_1 m_2}{a_1^2} C \sin(\lambda + \epsilon),$$

$$\frac{dn_2}{dt} = -\frac{3}{2}\frac{n_2}{a_2}\frac{da_2}{dt} = \frac{3\mathfrak{a}_2 m_1}{a_2^2} C \sin(\lambda + \epsilon).$$

Further, the total differentation of (11), § 28, with respect to t gives

$$\frac{d^2\zeta_1}{dt^2} = \frac{dn_1}{dt} + k_1, \qquad \frac{d^2\zeta_2}{dt^2} = \frac{dn_2}{dt^2} + k_2,$$

where k_1 and k_2 are of the second order with respect to the disturbing masses. Neglecting these (and always with limitation to periodic terms of the perturbing function depending on λ), we get

$$(12) \qquad \begin{cases} \dfrac{d^2\zeta_1}{dt^2} = \dfrac{3\mathfrak{a}_1 m_2}{a_1^2} C \sin(\lambda + \epsilon), \\[2ex] \dfrac{d^2\zeta_2}{dt^2} = \dfrac{3\mathfrak{a}_2 m_1}{a_2^2} C \sin(\lambda + \epsilon), \end{cases}$$

and, therefore, since $\lambda = \mathfrak{a}_1 \zeta_1 + \mathfrak{a}_2 \zeta_2$,

$$(13) \qquad \frac{d^2\lambda}{dt^2} = 3\left(\frac{\mathfrak{a}_1^{\ 2} m_2}{a_1^{\ 2}} + \frac{\mathfrak{a}_2^{\ 2} m_1}{a_2^{\ 2}}\right) C \sin(\lambda + \varepsilon).$$

If we put

$$(14) \qquad \lambda + \varepsilon = \mathfrak{a}_1 \zeta_1 + \mathfrak{a}_2 \zeta_2 + \varepsilon = V,$$

$$(15) \qquad -3\left(\frac{\mathfrak{a}_1^{\ 2} m_2}{a_1^{\ 2}} + \frac{\mathfrak{a}_2^{\ 2} m_1}{a_2^{\ 2}}\right) C = p,$$

and neglecting $\frac{d^2\varepsilon}{dt^2}$, which is likewise of the second order with respect to the disturbing masses, equation (13) passes into

$$(16) \qquad \frac{d^2 V}{dt^2} = -p \sin V,$$

where p can always be taken positively, since, in the opposite case, only ε, therefore also V, needs to be increased by π.

Equation (16) is not exact; in the second member, it lacks the other periodic terms and the terms of the second degree with respect to the disturbing masses. Only the terms depending on V have been taken into account. If we take p as constant (it depends only on the elements aud changes very slowly), equation (16) is that for the vibration of a pendulum in a plane.

Assume, now, that for a certain instant $V = 0$. If $\frac{dV}{dt}$ is then positive, V is positive and, by (16), $\frac{d^2 V}{dt^2}$, is negative. $\frac{dV}{dt}$ therefore decreases. If $\frac{dV}{dt}$ is negative, V is negative and $\frac{d^2 V}{dt^2}$ is positive and the absolute value of $\frac{dV}{dt}$ still decreases. Equation (16) therefore shows that the tendency of both planets is to lessen the increase, positive or negative, of V.

If we take the first case, that in which $V = 0$ the velocity $\frac{dV}{dt}$ is positive. Then while V increases, $\frac{dV}{dt}$ decreases, and V grows more and more slowly as it becomes larger. If $\frac{dV}{dt}$ does not become 0, or if its value is too large for $V = 0$, then it

decreases only so long as $V < \pi$. From here V grows again with accelerated velocity from π to 2π, and so on. This case corresponds to the perfect circular pendulum and is found only among the heavenly bodies.

If $\frac{dV}{dt}$ becomes 0 for a definite angle V_0, then V returns, at first slowly and then more rapidly to 0, then it becomes negative, reaching finally a value $-V_0$. Then it turns back, and so on. This is the analogue of the oscillating pendulum.

When $\frac{dV}{dt}$ is negative, the same treatment gives the same two cases.

By multiplying (16) by $\frac{dV}{dt}$ and integrating, we get

$$\tag{17} \tfrac{1}{2}\left(\frac{dV}{dt}\right)^2 = c + p\cos V,$$

$$\frac{dV}{dt} = \sqrt{2c + 2p\cos V}.$$

If c is positive and greater than p, V increases without limit. But if c is negative, or at least less than p, then $\frac{dV}{dt}$ becomes 0 for a definite angle V_0 which is determined by the equation

$$\cos V_0 = -\frac{c}{p}$$

But

$$V = \mathfrak{a}_1 \zeta_1 + \mathfrak{a}_2 \zeta_2 + \varepsilon,$$

or, suppressing the terms depending on the disturbing masses,

$$\frac{dV}{dt} = \mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2,$$

whence

$$(\mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2)^2 = 2c + 2p\cos V.$$

If (n_1) and (n_2) are the values of n_1 and n_2 corresponding to $V = 0$, then

$$(\mathfrak{a}_1(n_1) + \mathfrak{a}_2(n_2))^2 = 2c + 2p,$$
$$2c = (\mathfrak{a}_1(n_1) + \mathfrak{a}_2(n_2))^2 - 2p,$$

and (17) becomes

$$\frac{dV}{dt} = \sqrt{(\mathfrak{a}_1(n_1) + \mathfrak{a}_2(n_2))^2 - 4p\sin^2\tfrac{1}{2}V},$$

and, if V oscillates,

$$(\mathfrak{a}_1(n_1) + \mathfrak{a}_2(n_2))^2 < 4p,$$

and, therefore,

$$(18) \qquad [\mathfrak{a}_1(n_1) + \mathfrak{a}_2(n_2)] < 2\sqrt{p}.$$

When (18) is fulfilled, V oscillates about 0, and hence $\mathfrak{a}_1\zeta_1 + \mathfrak{a}_2\zeta_2$ about $-\varepsilon$. Aside from these oscillations $\mathfrak{a}_1\zeta_1 + \mathfrak{a}_2\zeta_2$ remains constant and, if $[n_1]$ and $[n_2]$ are the mean values of the mean daily motions, it follows that

$$(19) \qquad \mathfrak{a}_1[n_1] + \mathfrak{a}_2[n_2] = 0, \quad [n_1]:[n_2] = \mathfrak{a}_2 : -\mathfrak{a}_1.$$

From this the following important conclusion results:

Let there be two planets with approximately commensurable periodic times, when the sun only affects them. If the approximation surpasses a limit defined by (18), *the mutual attraction of the two planets will cause the approximate commensurability to become exact.*

This limit is not reached for Jupiter and Saturn. Relatively small changes in their major axes would, as Laplace has shown in the *Mécanique céleste*, cause the limit to be passed, and we would then have the remarkable phenomenon, for the two largest planets of the solar system, that when one had revolved five times about the sun, the other would have completed exactly two revolutions.

An oscillation through a small angle about a mean value is called a libration in astronomy. The duration is determined by the complete elliptic integral

$$T = 4\int_0^{V_0} \frac{dV}{\sqrt{2c + 2g\cos V}}.$$

It is evident that, in this case, the term of the perturbing function which depends on V is no longer properly periodic, and that its influence is exerted in a totally different manner on

the elements, than is that of the other periodic terms. Laplace made some studies of this case in which he showed that, taking the extreme excursion V as small, all the elements take part in the libration.

The investigations which have been made in an exact commensurability, are, of course, founded on assumptions which are themselves not exactly fulfilled. The terms of the perturbing function which do not depend on V and those of higher degree for the disturbing masses are neglected. Nevertheless, the results are of permanent value because they show that even a close approximation to commensurability, existing in two periods would bring with it no danger to our planetary system, but would rather bind the planets in still closer bonds, as is shown by the entire coincidence between the times of revolution and axial rotation of the moon.

37. The Exactness of the Formulas for the Variation of the Elements.

In the preceding sections the elements have been put in the form

$$(1) \qquad \begin{cases} a=[a]+(a), & \pi+[\pi]+(\pi), \\ e=[e]+(e), & \Omega=[\Omega]+(\Omega), \\ i=[i]+(i), & \zeta=[\zeta]+(\zeta). \end{cases}$$

The first terms in the second members are the secular values of the elements and the second the periodic values and are proportional to the disturbing masses. Each of the latter is of the frequently employed form (11) § 27, in which the secular values are to be substituted for the elements $a, e, \ldots$ In this sense $(a), (e), \ldots$ are functions of $[a], [e], \ldots$ and hence the elements $a, e, \ldots$ are functions of $[a], [e], \ldots$

Formulas (1) are only approximate and equations (8), § 28, are not entirely satisfied by substitution of them. In consequence, certain differences remain which are now to receive attention.

If (1) are differentiated from the above point of view, it follows, when subscripts are introduced, that

$$
(2)\left\{
\begin{aligned}
\frac{da_1}{dt} &= \frac{d[a_1]}{dt} + \frac{d(a_1)}{dt} \\
&= \frac{d[a_1]}{dt} + \frac{\partial(a_1)}{\partial[\zeta_1]}[n_1] + \frac{\partial(a_1)}{\partial[\zeta_2]}[n_2] + \frac{\partial(a_1)}{\partial[\zeta_3]}[n_3] + \dots \\
&\quad + \left\{ \frac{\partial(a_1)}{\partial[a_1]}\frac{d[a_1]}{dt} + \frac{\partial(a_1)}{\partial[e_1]}\frac{d[e_1]}{dt} + \dots + \frac{\partial(a_1)}{\partial[\zeta_1]}\left(\frac{d[\zeta_1]}{dt} - [n_1]\right) \right\} \\
&\quad + \left\{ \frac{\partial(a_1)}{\partial[a_2]}\frac{d[a_2]}{dt} + \frac{\partial[a_1]}{\partial[e_2]}\frac{d[e_2]}{dt} + \dots + \frac{\partial(a_1)}{\partial[\zeta_2]}\left(\frac{d[\zeta_2]}{dt} - [n_2]\right) \right\} \\
&\quad + \dots \\
\frac{de_1}{dt} &= \frac{d[e_1]}{dt} + \frac{\partial(e_1)}{\partial[\zeta_1]}[n_1] + \frac{\partial(e_1)}{\partial[\zeta_2]}[n_2] + \frac{\partial(e_1)}{\partial[\zeta_3]}[n_3] + \dots \\
&\quad + \left\{ \frac{\partial(e_1)}{\partial[a_1]}\frac{\partial[a_1]}{dt} + \dots \qquad + \frac{\partial(e_1)}{\partial[\zeta_1]}\left(\frac{d[\zeta_1]}{dt} - [n_1]\right) \right\} \\
&\quad + \left\{ \frac{\partial(e_1)}{\partial[a_2]}\frac{d[a_2]}{dt} + \dots \right\} \\
&\quad + \dots \\
\frac{d\zeta_1}{dt} - n_1 &= \frac{d[\zeta_1]}{dt} - [n_1] - (n_1) + \frac{\partial(\zeta_1)}{\partial[\zeta_1]}[n_1] + \frac{\partial(\zeta_1)}{\partial[\zeta_2]}[n_2] + \dots \\
&\quad + \left\{ \frac{\partial(\zeta_1)}{\partial[a_1]}\frac{d[a_1]}{dt} + \frac{\partial(\zeta_1)}{\partial[e_1]}\frac{d[e_1]}{dt} + \dots + \frac{\partial(\zeta_1)}{\partial[\zeta_1]}\left(\frac{d[\zeta_1]}{dt} - [n_1]\right) \right\} \\
&\quad + \dots
\end{aligned}
\right.
$$

There are corresponding equations for the derivatives of the other elements. Taking the second members of (2), we will first bring the expression

$$
(3) \qquad \frac{d[a_1]}{dt} + \frac{\partial(a_1)}{\partial[\zeta_1]}[n_1] + \frac{\partial(a_2)}{\partial[\zeta_2]}[n_2] + \dots
$$

into a very simple form. First $\frac{d[a_1]}{dt} = 0$. Then if $K\genfrac{}{}{0pt}{}{\cos}{\sin}\,\mathfrak{a}_1\zeta_1 + \mathfrak{a}_2\zeta_2$ is a term of $\frac{da_1}{dt}$ arising from the periodic term of R_1, then by § 29, we have for the corresponding term in (a),

$$
\frac{K}{\mathfrak{a}_1 n_1 + \mathfrak{a}_2 n_2} \cdot \genfrac{}{}{0pt}{}{\sin}{-\cos}\left(\mathfrak{a}_1\zeta_1 + \mathfrak{a}_2\zeta_2\right).
$$

The corresponding term in $\frac{\partial a_1}{\partial[\zeta_1]}[n_1] + \frac{\partial a_1}{\partial[\zeta_2]}[n_2]$ is again $K\genfrac{}{}{0pt}{}{\cos}{\sin}\left(\mathfrak{a}_1\zeta_1 + \mathfrak{a}_2\zeta_2\right)$. The expression (3) is then equal to the sec-

ond member of the first equation (8), § 28, if the secular values are used for the elements. This amounts to the following:

Let the perturbing function R_1 be expressed in terms of the elements, or

$$R = R(a_1, e_1, \ldots, a_2, e_2, \ldots). \tag{4}$$

If we substitute the values of these elements from (1), we get

$$R = R(([a_1]+(a_1)), \ldots ([a_2]+(a_2)), \ldots). \tag{5}$$

Now put, for the elements, only their secular values and indicate the result of this substitution by $\overline{R}$. Then

$$\overline{R} = R([a_1], [e_1], \ldots [a_2], [e_2], \ldots). \tag{6}$$

Then (3) becomes

$$-2\sqrt{\frac{[a_1]}{\mu_1}}\frac{\partial \overline{R}_1}{\partial [\zeta_1]}.$$

Considering further the remaining terms in the second member of equation (2), it is seen that they are all of the second order with respect to the disturbing masses, because each one is formed by two factors, each proportional to these masses. Since the first factor is purely periodic and the second purely secular, the product is purely periodic. If the sum of all these purely periodic terms be represented by $\{\{a\}\}$, that is

$$\begin{aligned}\{\{a_1\}\} = {} & \frac{\partial(a_1)}{\partial[a_1]}\frac{d[a_1]}{dt} + \frac{\partial(a_1)}{\partial[e_1]}\frac{d[e_1]}{dt} + \ldots \\ & + \frac{\partial(a_1)}{\partial[\zeta_1]}\left(\frac{d[\zeta_1]}{dt} - [n_1]\right) \\ & + \frac{\partial(a_1)}{\partial[a_2]}\frac{d[a_2]}{dt} + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \frac{\partial(a_1)}{d[\zeta_2]}\left(\frac{d[\zeta_2]}{dt} - n_2]\right) + \ldots\end{aligned} \tag{7}$$

the first of equations (2), becomes

$$\frac{da_1}{dt} = 2\sqrt{\frac{[a_1]}{\mu}}\frac{\partial \overline{R}}{\partial[\zeta_1]} + \{\{a_1\}\}. \tag{8}$$

Likewise

$$\begin{aligned}\frac{de_1}{dt} = {} & -\frac{\sqrt{1-[e_1]^2}\left(1-\sqrt{1-[e_1]^2}\right)}{[e_1]\sqrt{[a_1]\mu_1}}\frac{\partial \overline{R}_1}{\partial[\zeta_1]} \\ & -\frac{1}{[e_1]}\sqrt{\frac{1-[e_1]^2}{[a_1]\mu_1}}\frac{\partial \overline{R}}{\partial[\pi_1]} + \{\{e_1\}\},\end{aligned} \tag{9}$$

where $\{\{e_1\}\}$ is obtained from (7) by writing (e_1) for (a_1).

We will now substitute the value (1) in the second members of equations (8), § 28. We get

$$2\sqrt{\frac{a_1}{\mu_1}}\frac{\partial R_1}{\partial \zeta_1}=2\sqrt{\frac{[a_1]+(a_1)}{\mu_1}},\ \frac{\partial R_1(([a_1]+(a_1)),\ldots([a_2]+(a_2)),\ldots)}{\partial([\zeta_1]+(\zeta_1))},$$

The periodic terms (a_1), (e_1), ... are of the order of the disturbing masses. If the second member is developed in ascending powers of (a_1), (e_1), ... and limited to the first degree, then

$$2\sqrt{\frac{a_1}{\mu_1}}\frac{\partial R}{\partial \zeta_1}=2\sqrt{\frac{[a_1]}{\mu_1}}\frac{\partial \overline{R}_1}{\partial[\zeta_1]}+\frac{(a_1)}{\sqrt{\mu_1[a_1]}}\frac{\partial \overline{R}}{\partial[\zeta_1]}$$
$$+2\sqrt{\frac{[a_1]}{\mu}}\left(\frac{\partial^2 \overline{R}_1}{\partial[\zeta_1]\partial[a_1]}(a_1)+\frac{\partial^2 \overline{R}_1}{\partial[\zeta_1]\partial[e_1]}(e_1)+\ldots\right.$$
$$\left.+\frac{\partial^2 \overline{R}_1}{\partial[\zeta_1]\partial[a_2]}(a_2)+\quad\ldots\ldots\ldots\right).$$

If, for brevity, we now put

$$(10)\qquad [\{a_1\}]=\frac{(a_1)\partial \overline{R}_1}{\sqrt{\mu_1[a_1]}\partial[\zeta_1]}+2\sqrt{\frac{[a_1]}{\mu_1}}\left(\frac{\partial^2 \overline{R}_1}{\partial[\zeta_1]\partial[a_1]}(a_1)+\ldots\right),$$

where $[\{a_1\}]$ is of the second degree with reference to the disturbing masses.

The preceding equation now becomes

$$(11)\qquad 2\sqrt{\frac{a_1}{\mu_1}}\frac{\partial R_1}{\partial \zeta_1}=2\sqrt{\frac{[a_1]}{\mu_1}}\frac{\partial \overline{R}_1}{\partial[\zeta_1]}+[\{a_1\}].$$

If (1) is substituted in the first member of the first of equations (8), § 28, the form (8) results; if in the second member, the form (11). The difference between the two members is

$$\{\{a_1\}\}-[\{a_1\}]$$

which is of the second degree with respect to the disturbing masses.

In the same way it may be shown that the other equations (8), § 28 (taking instead of the third, equation (11), § 28), are likewise of the second degree with respect to the disturbing masses. The differences are, in general, very small and the

accuracy of (1) is thus shown. But as the small term in a differential equation may, by integration, eventually have an appreciable effect, we will occupy ourselves in the next sections in defining this effect more closely.

38. The Improvement of the Theory of Variation of Constants by Including Terms Depending on the Second Powers of the Masses.

In the preceding sections the elements and coordinates have been expressed in terms of the time t and $6n$ arbitrary constants. The latter are

(1)
1. The n secular values of the mean distances,
2. The $2n$ arbitrary angles δ and δ' in the secular arguments G and Γ,
3. The $2n$ arbitrary factors K and K' in the expressions of the eccentricities and inclinations,
4. The n arbitrary angles ϵ in the mean longitudes ζ.

These expressions for the elements do not entirely satisfy equations (8), § 28, but there remain residuals of the second degree in the masses. The best way to make these residuals disappear is to turn again to the method of variation of constants. Let the quantities in (1) be no longer considered as constants but as functions of the time, whose determination shall be such that the substitution of their expressions in equations (8), § 28, shall render them identical. For instance, it will make

$$\frac{da_1}{dt} - 2\sqrt{\frac{a_1}{\mu_1}}\frac{\partial R_1}{\partial \zeta_1} = 0, \tag{2}$$

where, in the formation of $\dfrac{da_1}{dt}$, it is to be remembered that a_1 contains the time not only explicitly, but also implicitly, in so far as quantities appear in a_1 which were constants in (1) but are now functions of t. If the partial derivatives of a_1 with respect to t, in so far as t is explicit in a_1 are now denoted by $\dfrac{\partial a_1}{\partial t}$ we get by the preceding section

(3) $$\frac{\partial a_1}{\partial t}-2\sqrt{\frac{a_1}{\mu_1}}\frac{\partial R_1}{\partial \zeta_1}=\{\{a_1\}\}-[\{a_1\}].$$

By subtracting (2) from (3), it follows that

(4) $$\frac{da_1}{dt}-\frac{\partial a_1}{\partial t}=\{\{a_1\}\}-[\{a_1\}].$$

Similarly

(5) $$\frac{de_1}{dt}-\frac{\partial e}{\partial t}=\{\{e_1\}\}-[\{e_1\}], \text{ etc.}$$

The second members of these equations are functions of t and of the variables given in (1). The first members contain the parts of $\frac{da}{dt}:\frac{de}{dt},\ldots$ which remain when the explicit derivatives with respect to t are removed and they therefore, vary only for the quantities (1). From the $6n$ equations (4) and (5), the derivatives of the quantities (1) can be obtained. Yet, with close approximation the secular values of the elements may be substituted in second members of (4) and (5) for the elements themselves. The equation (4), for instance, may be written, by (1), § 37,

$$\frac{d[a_1]}{dt}-\frac{\partial[a_1]}{\partial t}+\frac{d(a_1)}{dt}-\frac{\partial(a_1)}{\partial t}=\{\{a_1\}\}-[\{a_1\}].$$

Now (a_1) is of the first degree with reference to the disturbing masses, and hence $\frac{d(a_1)}{dt}-\frac{\partial(a_1)}{\partial t}$ is of the third degree since the derivatives of the former constants (1) are, by (4) and (5), of the second degree. Neglecting terms of the third degree, we get, therefore,

(6) $$\frac{d[a_1]}{dt}-\frac{\partial[a_1]}{\partial t}=\{\{a_1\}\}-[\{a_1\}],$$

and similarly

(7) $$\frac{d[e_1]}{dt}-\frac{\partial[e_1]}{\partial t}=\{\{e_1\}\}-[\{e_1\}], \text{ etc.}$$

Equation (6) is in exactly the form taken by the derivative of a previous constant, namely $[a_1]$. It is $\frac{\partial[a_1]}{\partial t}=0$, and hence

$$(8) \qquad \frac{d[a_1]}{dt} = \{\{a_1\}\} - [\{a_1\}].$$

Designating the constants of (1) in order by $p_1, p_2, \ldots p_{6n}$, we have

$$\frac{d[e_1]}{dt} - \frac{\partial[e_1]}{\partial t} = \frac{\partial[e_1]}{\partial p_1}\frac{dp_1}{dt} + \frac{\partial[e_1]}{\partial p_2}\frac{dp_2}{dt} + \ldots + \frac{\partial[e_1]}{\partial p_{6n}}\frac{dp_{6n}}{dt},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and (7) becomes

$$(9) \quad \frac{\partial[e_1]}{\partial p_1}\frac{dp_1}{dt} + \frac{\partial[e_1]}{\partial p_2}\frac{dp_2}{dt} + \ldots + \frac{\partial[e_1]}{\partial p_{6n}}\frac{dp_{6n}}{dt} = \{\{e_1\}\} - [\{e_1\}], \text{ etc.}$$

From (9) the derivatives of the p's can be obtained.

It is not the purpose here to complete the determination but rather to make some general remarks on it. The coefficients $\frac{\partial[e_1]}{\partial p_1}, \frac{\partial[e_1]}{\partial p_2}, \ldots$ of the equations (9) contain only secular terms. The $\frac{dp_1}{dt}, \ldots$ will, therefore, contain only secular terms if such occur in $[\{a_1\}], [\{e_1\}], \ldots$ (We know from the preceding section that they do not occur in $\{\{a_1\}\}$, etc.).

In accordance with the principles of § 29, we will here also limit the consideration to the secular terms in the second members of (8) and (9), that is to those independent of ζ, and we will pass on later to the periodic members which will be treated by the methods used before.

The next section will show that there are no secular terms in $[\{a_1\}]$ and that with limitation to these, equation (8) becomes

$$\frac{d[a_1]}{dt} = 0,$$

so that the previous constants $[a]$ still remain constant. With this determined, the remaining equations (8) can be approximately integrated by the previous methods. I will not enter into the minuter details, but will only note that K and K' also remain constant, if, to the eccentricities and inclinations other secular terms are added, whose arguments are formed by integral combinations of G and Γ and whose cofficients contain the disturbing masses as factors. Also g' and γ' are increased by small fractions, which are likewise of the first power of the masses.

39. The Invariability of the Major Axes.

Poisson proved that the major axes are invariable even when, as in the preceding sections, account is taken of terms of the second degree with reference to the disturbing masses. (See Poisson's paper *Sur les inégalités séculaires des moyens mouvements des planètes. Journal de l'Ecole polytechnique, Tome VIII, 1809*). In what follows, Laplace's demonstration is used, though in a modified form.

After the preceding section, it remains only to be shown that in

$$(1) \qquad [\{a_1\}]$$

there are no terms independent of the mean longitudes. Representing, since there is now no chance for confusion, the $[a]$, $[e], \ldots$ by $a, e, \ldots$ and also $\bar{R}$ by R, the quantity $[\{a_1\}]$ consists, by (10), § 37, of terms of the three following forms:

$$(2) \quad \begin{cases} (\mathrm{I}) \ (a)\dfrac{\partial R_1}{\partial \zeta_1}, \\ (\mathrm{II}) \ \dfrac{\partial^2 R_1}{\partial \zeta_1 \partial a_1}(a_1) + \dfrac{\partial^2 R_1}{\partial \zeta_1 \partial e_1}(e_1) + \ldots + \dfrac{\partial^2 R_1}{\partial \zeta_1 \partial \zeta_1}(\zeta_1), \\ (\mathrm{III}) \ \dfrac{\partial^2 R_1}{\partial \zeta_1 \partial a_2}(a_2) + \dfrac{\partial^2 R_1}{\partial \zeta_1 \partial e_2}(e_2) + \ldots + \dfrac{\partial^2 R_1}{\partial \zeta_1 \partial \zeta_2}(\zeta_2). \end{cases}$$

Each term in (2) is a product of two factors. R is here taken as developed in the usual manner, as in form (49), § 24. The quantities (a), $(e), \ldots$ are of the same form. If the factors are multiplied out term by term and use is made of the formula

$$\cos\alpha\cos\beta = \tfrac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta)),$$

it becomes clear that there are only two such terms that can give secular terms, both dependent on the same argument

$$(3) \qquad \lambda = \mathfrak{a}_1\zeta_1 + \mathfrak{a}_2\zeta_2$$

in which both integral numbers $\mathfrak{a}_1$ and $\mathfrak{a}_2$ must not be 0, at the same time, because only periodic terms appear in $(a), \ldots$ Let the term in R, corresponding to this argument, be

$$(4) \qquad k\sin\lambda + l\cos\lambda.$$

Then the corresponding terms in $(a), (e), \ldots$ are to be formed according to §29, and substituted in (2).

Take first the term I of (2). The term in (a_1) depending on λ is by (4), §33

$$=2\sqrt{\frac{a_1}{\mu_1}}\frac{\mathfrak{a}_1}{\mathfrak{a}_1 n_1+\mathfrak{a}_2 n_2}(k\sin\lambda+l\cos\lambda),$$

and in $\dfrac{\partial R}{\partial \zeta_1}$

$$\mathfrak{a}_1(k\cos\lambda-l\sin\lambda).$$

By multiplication the purely periodic term

$$2\sqrt{\frac{a_1}{\mu_1}}\frac{\mathfrak{a}_1^2}{\mathfrak{a}_1 n_1+\mathfrak{a}_2 n_2}\left(\frac{k^2-l^2}{2}\sin 2\lambda+kl\cos 2\lambda\right)$$

is formed in (I).

Take the sum II next. If the first term in the expression for (ζ)

$$-\frac{3}{a_1^2}\frac{\mathfrak{a}_1}{(\mathfrak{a}_1 n_1+\mathfrak{a}_2 n_2)^2}(-k\cos\lambda+l\sin\lambda)$$

is disregarded, the products of the terms in II, which depend on the same argument λ, all vanish. On the other hand the product of this term with $\dfrac{\partial^2 R}{\partial\zeta_1\partial\zeta_2}$ is

$$\frac{3\mathfrak{a}_1^3}{a^2(\mathfrak{a}_1 n_1+\mathfrak{a}_2 n_2)^2}(k\sin\lambda+l\cos\lambda)(-k\cos\lambda+l\sin\lambda)$$
$$=-\frac{3\mathfrak{a}_1^3}{a_1^2(\mathfrak{a}_1 n_1+\mathfrak{a}_2 n_2)^2}\left(\frac{k^2-l^2}{2}\sin 2\lambda+kl\cos 2\lambda\right),$$

and, hence, also a periodic term.

The treatment of III is somewhat more difficult. Since R_1 is here differentiated with respect to the elements of the planet with subscript 2, the part of R_1 involved is

$$m_2\left(\frac{1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}-\frac{x_1x_2+y_1y_2+z_1z_2}{r_2^3}\right),$$

and since R_1 is also differentiated with respect to ζ_1 and only such values of λ can enter as are dependent on ζ_1, the terms in $(a_2), (e_2), \ldots$ depending on ζ_1 must also be dropped. $(a_2), (e_2), \ldots$ arise from R_2 and in this we have then to take into account only the part

$$m_1\left(\frac{1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}-\frac{x_1x_2+y_1y_2+z_1z_2}{r_1^3}\right).$$

Let

(5) $$\frac{1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}=\Sigma(P\sin\lambda+Q\cos\lambda),$$

(6) $$x_1x_2+y_1y_2+z_1z_2=\Sigma(p\sin\lambda+q\cos\lambda).$$

The factors m_1 and m_2 in R_1 and R_2 may be omitted as immaterial in the present treatment. If now, R_1^λ and R_2^λ represent the terms dependent on the argument λ, then, by (7) and (8), § 36,

(7) $$R_1^\lambda=\left(P-\frac{\mathfrak{a}_2^2n_2^2}{\mu_2}p\right)\sin\lambda+\left(Q-\frac{\mathfrak{a}_2^2n_2^2}{\mu_2}q\right)\cos\lambda,$$

(8) $$R_2^\lambda=\left(P-\frac{\mathfrak{a}_1^2n_1^2}{\mu_1}p\right)\sin\lambda+\left(Q-\frac{\mathfrak{a}_1^2n_1^2}{\mu_1}q\right)\cos\lambda.$$

If $R_1^\lambda=R_2^\lambda$, no secular terms will arise in III. The proof of this is made exactly as for II, and in the formation of (a_2), (e_2), . . . the consideration may be limited to the difference

(9) $$V^\lambda=R_2^\lambda-R_1^\lambda=-\left(\frac{\mathfrak{a}_1^2n_1^2}{\mu_1}-\frac{\mathfrak{a}_2^2n_2^2}{\mu_2}\right)(p\sin\lambda+q\cos\lambda).$$

Turning next to the elements as they are contained in (7) and (9), aside from the coefficients P, Q, p, q, it appears that the only ones occurring in n_1 and n_2 are a_1 and a_2. Under limitation to these we have in $\frac{\partial^2R_1}{\partial\zeta_1\partial a_2}$ the term

(10) $$-\frac{3}{a_2^4}\mathfrak{a}_1\mathfrak{a}_2^2(p\cos\lambda-q\sin\lambda).$$

The corresponding term in (a_2) is by (4), § 33,

(11) $$2\sqrt{\frac{a_2}{\mu_2}}\left(\frac{\mathfrak{a}_1^2n_1^2}{\mu_1}-\frac{\mathfrak{a}_2^2n_2^2}{\mu_2}\right)\frac{\mathfrak{a}_2}{(\mathfrak{a}_1n_1+\mathfrak{a}_2n_2)}(p\sin\lambda+q\cos\lambda).$$

The product of (10) and (11) gives only periodic terms.

It is further to be observed that a_2 which occurs in (9) is contained in n_2^2. And its product with

$$\frac{\mathfrak{a}_1\mathfrak{a}_2^3n_2^2}{\mu_2}(p\sin\lambda+q\cos\lambda),$$

the part of $\frac{\partial^2R_1}{\partial\zeta_1\partial\zeta_2}$ which arises from the term

$$-\frac{\mathfrak{a}_2^2 n_2^2}{\mu_2}(p\sin\lambda+q\cos\lambda)$$

of R_1, likewise gives only periodic terms.

Hence, only the following suppositions can be made:

First, R_1^λ is limited to $(P\sin\lambda+Q\cos\lambda)$ and V^λ retains its value (9).

Second, R_1^λ and V^λ are both limited to $(p\sin\lambda+q\cos\lambda)$.

The second supposition affords only periodic terms which follow exactly as for II. It remains to investigate only the first supposition, namely that in which only the term

$$\text{(12)}\qquad \frac{1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}=\Sigma(P\sin\lambda+Q\cos\lambda)$$

remains in R_1 and only the term

$$\text{(13)}\qquad \sum\left(\frac{\mathfrak{a}_1^2 n_1^2}{\mu_1}-\frac{\mathfrak{a}_2^2 n_2^2}{\mu_2}\right)(p\sin\lambda+q\cos\lambda)$$
$$=(x_1x_2+y_1y_2+z_1z_2)\left(\frac{1}{r_2^3}-\frac{1}{r_1^3}\right)$$

in R_2.

The value of R_2 can be put in another form. For undisturbed motion it is

$$R_2=(x_1x_2+y_1y_2+z_1z_2)\left(\frac{1}{r_2^3}-\frac{1}{r_1^3}\right)$$
$$=-\frac{x_1d^2x_2+y_1d^2y_2+z_1d^2z_2}{\mu_2dt^2}+\frac{x_2d^2x_1+y_2d^2y_1+z_2d^2z_1}{\mu_1dt^2}.$$

In this case it is permissible to put $\mu_1=\mu_2$, since their difference depends only on the disturbing masses. The above equation then becomes

$$R_2=(x_1x_2+y_1y_2+z_1z_2)\left(\frac{1}{r_2^3}-\frac{1}{r_1^3}\right)$$
$$=-\frac{1}{\mu_1}\frac{d}{dt}(x_1x_2'-x_2x_1'+y_1y_2'-y_2y_1'+z_1z_2'-z_2z_1').$$

If we put

$$\text{(14)}\qquad W=\int R_2\,dt$$

then

$$\text{(15)}\qquad W=-\frac{1}{\mu_1}(x_1x_2'-x_2x_1'+y_1y_2'-y_2y_1'+z_1z_2'-z_2z_1').$$

This integral of the perturbing function must be introduced into the expressions for the periodic terms. Representing, as before, Poisson's expressions by (a, a), (a, e), ... we have

$$(16)\qquad \begin{cases} (a_2) = (a_2, e_2)\dfrac{\partial W}{\partial e_2} + (a_2, \varepsilon_2)\dfrac{\partial W}{\partial \zeta_2} + \ldots \\ (e_2) = (e_2, a_2)\dfrac{\partial W}{\partial a_2} + (e_2, \varepsilon_2)\dfrac{\partial W}{\partial \zeta_2} + \ldots \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \end{cases}$$

Finally after an easy reduction, which is due to the differentiation of n_2 (which contains a_2 and by (15) appears in x_2', y_2', z_2'), with respect to a_2 we get

$$(17)\qquad (\zeta_2) = (\varepsilon_2, a_2)\frac{\partial W}{\partial a_2} + (\varepsilon_2, e_2)\frac{\partial W}{\partial e_2} + \ldots$$

Introducing this into III and for brevity putting

$$(18)\qquad \frac{\partial R_1}{\partial \zeta_1} = V,$$

the expression III becomes

$$(19)\qquad \begin{aligned} &(a_2, e_2)\left(\frac{\partial V}{\partial a_2}\frac{\partial W}{\partial e_2} - \frac{\partial V}{\partial e_2}\frac{\partial W}{\partial a_2}\right) \\ &+ (a_2, \varepsilon_2)\left(\frac{\partial V}{\partial a_2}\frac{\partial W}{\partial \varepsilon_2} - \frac{\partial V}{\partial \varepsilon_2}\frac{\partial W}{\partial a_2}\right) \\ &+ \ldots\ldots\ldots \\ &+ \ldots\ldots\ldots \end{aligned}$$

After the coordinates and component velocities have been introduced into this expression, it can, by formula (2), § 12, be changed into

$$(20)\qquad \frac{\partial V}{\partial x_2}\frac{\partial W}{\partial x_2'} - \frac{\partial V}{\partial x_2'}\frac{\partial W}{\partial x_2} + \frac{\partial V}{\partial y_2}\frac{\partial W}{\partial y_2'} - \frac{\partial V}{\partial y_2'}\frac{\partial W}{\partial y_2} + \frac{\partial V}{\partial z_2}\frac{\partial W}{\partial z_2'} - \frac{\partial V}{\partial z_2'}\frac{\partial W}{\partial z_2}.$$

It only remains to prove that there is no secular term in (20). This can be done as follows. Omitting the factor μ, we have

$$(21)\qquad W = x_1x_2' - x_2x_1' + y_1y_2' - y_2y_1' + z_1z_2' - z_2z_1',$$

and hence, (20) becomes

(22) $$\frac{\partial V}{\partial x_2}x_1+\frac{\partial V}{\partial y_2}y_1+\frac{\partial V}{\partial z_2}z_1+\frac{\partial V}{\partial x_2'}x_1'+\frac{\partial V}{\partial y_2'}y_1'+\frac{\partial V}{\partial z_2'}z_1'.$$

The quantity V from (18) must now be substituted in (22). We have

$$R_1=\frac{1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}=\frac{1}{r_{12}},$$

and therefore,

(23) $$V=\frac{\partial R_1}{\partial \zeta_1}=\frac{\partial R_1}{\partial x_1}\frac{\partial x_1}{\partial \zeta_1}+\frac{\partial R_1}{\partial y_1}\frac{\partial y_1}{\partial \zeta_1}+\frac{\partial R_1}{\partial z_1}\frac{\partial z_1}{\partial \zeta_1}$$
$$=\frac{1}{n_1}\left(\frac{\partial R_1}{\partial x_1}x_1'+\frac{\partial R_1}{\partial y_1}y_1'+\frac{\partial R_1}{\partial z_1}z_1'\right).$$

Consequently

$$\frac{\partial V}{\partial x_2'}=\frac{\partial V}{\partial y_2'}=\frac{\partial V}{\partial z_2'}=0,$$

and (22) reduces to

(24) $$\frac{\partial V}{\partial x_2}x_1+\frac{\partial V}{\partial y_2}y_1+\frac{\partial V}{\partial z_2}z_1.$$

This expression can now be so changed that the absence of secular terms is at once visible. V is, by (23), a homogeneous function of degree -2 in $x_1, y_1, z_1, x_2, y_2, z_2$, and depends only on (x_1-x_2), (y_1-y_2), (z_1-z_2). Hence

$$\frac{\partial V}{\partial x_1}=\frac{\partial V}{\partial x_2},\ldots,$$

and

$$\frac{\partial V}{\partial x_2}(x_2-x_1)+\frac{\partial V}{\partial y_2}(y_2-y_1)+\frac{\partial V}{\partial z_2}(z_2-z_1)=-2V,$$

and (24) becomes

(25) $$2V+\frac{\partial V}{\partial x_2}x_2+\frac{\partial V}{\partial y_2}y_2+\frac{\partial V}{\partial z_2}z_2.$$

$V=\frac{\partial R_1}{\partial \zeta_1}$ contains no term independent of ζ_1. Hence no such term can appear in (25), for after substituting for V its value (18), we can at once write (25) as follows:

(26) $$\frac{\partial}{\partial \zeta_1}\left(2R_1+x_2\frac{\partial R_1}{\partial x_2}+y_2\frac{\partial R_1}{\partial y_2}+z_2\frac{\partial R_1}{\partial z_2}\right).$$

Therewith the proof is completed that $\frac{da}{dt}$ can contain no secular term even when the second powers of the masses are taken into account, and we can now, with greater emphasis than in §30, state the proposition

Excepting periodic perturbations the major axes remain constant.

40. The Form in which the Elements and Coordinates appear as Functions of the Time.

In the preceding sections, the elements a, h, l, p, q, ζ of the planets have been represented by aid of a definite number of angles, all of which are linear functions of the time. They are

(1)
1. The n secular values $[\zeta_1], [\zeta_2], \ldots [\zeta_n]$ of the mean longitudes.
2. The n angles $G_1, G_2, \ldots G_n$.
3. The $(n-1)$ angles $\Gamma_1, \Gamma_2, \ldots \Gamma_{n-1}$.

We will now disregard the manner in which these angles have been introduced (ζ for instance, in the first approximation to elliptic motion and G and Γ through secular perturbations). We will also disregard the fact that the coefficients of the time in G and Γ are very much smaller than in ζ and we will regard the above angles as, so to speak, of equal value. Designating them in order by

$$l_1, l_2, l_3, \ldots l_{3n-1}, \tag{2}$$

it appears that any angle l_λ is of the form

$$l_\lambda = a_\lambda t + b_\lambda, \tag{3}$$

where a_λ and b_λ are constants.

The x coordinates are expressed by an infinite series of the form

$$x = \Sigma k \cos(\mathfrak{a}_1 l_1 + \mathfrak{a}_2 l_2 + \ldots + \mathfrak{a}_{3n-1} l_{3n-1}), \tag{4}$$

where the numbers $\mathfrak{a}_1, \mathfrak{a}_2, \ldots$ can take all values which satisfy the condition

$$\mathfrak{a}_1 + \mathfrak{a}_2 + \ldots \mathfrak{a}_{3n-1} = 1. \tag{5}$$

Further

(6) $$y = \Sigma k \sin(\mathfrak{a}_1 l_1 + \mathfrak{a}_2 l_2 + \ldots + \mathfrak{a}_{3n-1} l_{3n-1})$$

with the same coefficients k.

Finally

(7) $$z = \Sigma k' \sin(\mathfrak{a}_1' l_1 + \mathfrak{a}_2' l_2 + \ldots + \mathfrak{a}_{3n-1}' l_{3n-1})$$

where, however,

(8) $$\mathfrak{a}_1' + \mathfrak{a}_2' + \ldots + \mathfrak{a}_{3n-1}' = 0.$$

The invariable plane is taken as the plane xy. The k's and k''s are functions of only $(3n-1)$ constants, namely,

(9) $$\begin{cases} 1.\ \text{The } n \text{ secular values of the mean distances } [a_1], [a_2] \ldots, \\ 2.\ \text{The } n \text{ constants of integration } K_1, \ldots K_n, \\ 3.\ \text{The } (n-1) \text{ constants of integration, } K_1', \ldots K_{n-1}'. \end{cases}$$

The $(3n-1)$ coefficients a_λ in the l's are, by §§31 and 32 likewise functions of these $(3n-1)$ constants (9). The constants can, therefore, be expressed by the quantities a_λ and we may even say that the coefficients k are functions of a_λ. The quantities b_λ are constants additive to the l's and do not effect the coefficients k.

In the forms (4), (6) and (8) it is especially noteworthy that the time appears only in periodic form and that all the angles are linear functions of the time. It appears from this that each term, in so far as it is not constant, returns to the same value in equal intervals of time, and that its course is shown by a regular wavy line. Our planetary system must turn about its center of gravity with the greatest regularity, but periodic motions must be innumerable and their periods and magnitudes must show the greatest diversity. There is no initial condition in the past from which it has gradually developed, nor is a final condition necessary towards which it tends.

It is hardly necessary to warn the reader that these remarks are of force only when all other agents are excluded except gravitation. In fact there are others, but the duration of active observation is not more than enough to afford a slightest trace of them. It is not difficult to analyze their action, but

they are not included in the purpose of this work which is devoted solely to the study of motions resulting from gravitation.

Yet, with this limitation, the doctrine of the eternal stability of the solar system is very far from standing on the foundation of certainty due to perfectly rigorous deduction. No pains have been spared in bringing clearly into view the terms omitted in the integration of the differential equations, and no one has succeeded in showing that the general form of the expressions means more than that it is an approximation good for so many thousands of years, and so near the truth that it represents very approximately nearly all observations, in so far as the constants of integration have been suitably determined.

On the other hand there is a circumstance which continually warns us against a too early celebration of success. It is the reappearance of that denominator which is summoned by the integration of terms of the form $\frac{\sin}{\cos}(\alpha t + \beta)$. The most able astronomers and mathematicians have given it their care. Yet, notwithstanding the keenness of their investigations, including as they do those of Laplace, given in § 36, the case has not been completely analyzed. It is known, to be sure, that this denominator can only be great or infinite in our system when the coefficient of the respective term is very small, yet this does not at all dispose of the doubt which has recently led to the investigations of Professor Gylden.

When the terms referred to are neglected in the differential equation this part is eliminated. Yet it is certain that the neglected terms will have a sensible influence in the course of thousands of years, and in any case, by their neglect the solution is not rigorous, and not good for all time. The way in which the matter stands at present may be stated as follows:

The equilibrium of our planetary system has, by the admirable labors of the mathematicians, been shown to be true for periods of time which are very long, viewed from a human standpoint, but it has not been proven to exist forever.

The forms (4) to (8) for the coordinates were reached as an approximate result from the theory of variation of elements

by means of certain circumstances,—the smallness of the perturbing masses, and the smallness of the eccentricities and inclinations. If this represents the facts, is it because it expresses a mathematical law, and not because it is merely an interpolation form? According to my idea, this question is of fundamental significance, for an affirmative answer to it would give a brilliant justification to the acute researches of Laplace and Lagrange. It has been made a matter of reproach to them that their methods of integration lacked mathematical rigor and it has caused a feeling of uncertainty with a regard to neglected terms, a feeling which may be fallacious, and which, in fact, has proven fallacious in one case, as shown in § 26. Many mathematicians, led by the facts, have hesitated to admit the value of their results. But they undoubtedly felt, more strongly than others, all scruples which could be raised, and if any improvement had been possible, they would have made it. If we succeed in proving in any other way the laws which they discovered by imperfect deduction, to them belongs the great service of having pursued, undisturbed by doubts, and of having finally solved the great problem of the motions of heavenly bodies submitted to Newton's law of gravitation.

This other way so far as the anthor knows, is as yet hardly begun. Certain formal results have been obtained, but the question of convergence has not been taken up.

Forms (4) to (8) are apparently quite general for they hold without the limitation to small masses relative to the controlling mass, and to small eccentricities and inclinations. The preponderance of the mass of one point does not guarantee the stability when the latter is taken from the general standpoint already defined. If stability means a peculiarity of the system by virtue of which all the distances of the bodies from each other are always finite (never becoming 0 or infinite) this can be fulfilled, however large the masses may be, if the initial conditions have been suitably chosen. With this generalized conception of the problem, it is plain that we cannot take refuge either in the method of absolute perturbations nor in that of variation of elements. Hence it seems to me that forms (4) and (8)

form a guiding line for further investigations, that the above mentioned theories are to be abandoned after they have been exhausted and that the problem of n bodies, in its general form, should be undertaken entirely independently without recourse to Kepler's laws or other intermediary as initial points.

That astronomers have to-day reached this point is shown by the fact that the following offer of a prize, published in *Acta Mathematica* in 1885. The proposal is as follows:

"Any system of material points subject to Newton's law being given, with the conditions that there shall be no collisions, to develop the coordinates of each point in an infinite series of terms composed of known functions of the time, which series shall converge uniformly for a period of unlimited duration.

"That the solution of this problem, which must afford us in its course an insight into the most important of the phenomena of our system is not only possible, but is attainable with the analytical helps already existing, is shown by the statement of Lejeune-Dirichlet,* who, just before his death, confided to a mathematical friend that he had found a general method for the integration of the differential equations of mechanics, and that by the application of this method, he had succeeded in rigorously proving the stability of our system. Unfortunately we know nothing of Dirichlet's method except the indication that the small oscillations afforded him a foothold in the discovery. It may be taken as certain that it does not consist in difficult and complicated analysis, but in the introduction of a simple fundamental idea, the rediscovery of which may follow an earnest and persistent search.

"If the problem offers difficulties which cannot be overcome for the time, the prize can be given for a work in which any other important problem in mechanics is completely solved in the manner above indicated."

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* Kummer, Gedächtnissrede auf Lejeune-Dirichlet. Abhandlungen der Königl. Akademie der Wissenschaftenzu Berlin 1868, p. 35.

41. Several General Formulas Relating to the Coefficients in the Development of the Coordinates in Trigonometrical Series.

In §§ 36, 26 and 27, the coordinates are developed with reference to the invariable plane (or rather a plane parallel to it and passing through the sun) in the following form

$$(1)\quad \begin{cases} x_\lambda = \Sigma K_\lambda(\mathfrak{a}_1, \mathfrak{a}_2, \ldots \mathfrak{a}_{3n-1}) \cos(\mathfrak{a}_1 l_1 + \mathfrak{a}_2 l_2 + \ldots + \mathfrak{a}_{3n-1} l_{3n-1}) \\ \quad = \Sigma K_\lambda \cos H, \\ y_\lambda = \Sigma K_\lambda(\mathfrak{a}_1, \mathfrak{a}_2, \ldots \mathfrak{a}_{3n-1}) \sin(\mathfrak{a}_1 l_1 + \mathfrak{a}_2 l_2 + \ldots + \mathfrak{a}_{3n-1} l_{3n-1}) \\ \quad = \Sigma K_\lambda \sin H, \\ z_\lambda = \Sigma k_\lambda(\mathfrak{b}_1, \mathfrak{b}_2 \ldots \mathfrak{b}_{3n-1}) \sin(\mathfrak{b}_1 l_1 + \mathfrak{b}_2 l_2 + \ldots + \mathfrak{b}_{3n-1} l_{3n-1}) \\ \quad = \Sigma k_\lambda \sin h, \end{cases}$$

in which it is to be noted that

a. In the first and second formulas the Σ relates to all positive and negative values of the integral numbers $\mathfrak{a}_1, \ldots \mathfrak{a}_{3n-1}$ which satisfy the condition

$$(2)\qquad \mathfrak{a}_1 + \mathfrak{a}_2 + \ldots + \mathfrak{a}_{3n-1} = 1.$$

b. In the third the numbers $\mathfrak{b}$ are such that

$$(3)\qquad \mathfrak{b}_1 + \mathfrak{b}_2 + \ldots + \mathfrak{b}_{3n-1} = 0.$$

c. The $K_\lambda(\mathfrak{a}_1, \ldots \mathfrak{a}_{3n-1})$ and also $k_\lambda(\mathfrak{b}_1, \mathfrak{b}_2, \ldots \mathfrak{b}_{3n-1})$ are constant coefficients.

d. Each angle l_λ is of the form

$$(4)\qquad l_\lambda = a_\lambda t + \beta_\lambda.$$

In order that the coefficients k_λ, may be single-valued we will assume

$$(5)\qquad k_\lambda = (-\mathfrak{b}_1, -\mathfrak{b}_2, \ldots -\mathfrak{b}_{3n-1}) = -k_\lambda(\mathfrak{b}_1, \mathfrak{b}_3, \ldots \mathfrak{b}_{3n-1}).$$

Let it be assumed that equations (1) are not simply approximate, as was done in the preceding section, but that they are rigorous. Also let it be assumed that (1) are unconditionally convergent, and, therefore, that the sums of the absolute values of the coefficients

$$(6)\qquad \Sigma[K_\lambda],\ \Sigma[k_\lambda]$$

are likewise convergent.

The terms in (1) will naturally be arranged in the order of the integral numbers $\mathfrak{a}$ and $\mathfrak{b}$.

The angles l are all linear functions of the time, and the same must be true of H and h. Let

$$(7) \qquad H = Nt + \Delta, \quad h = nt + \delta,$$

where

$$(8) \qquad N = \mathfrak{a}_1\alpha_1 + \mathfrak{a}_2\alpha_2 + \ldots, \quad \Delta = \mathfrak{a}_1\beta_1 + \mathfrak{a}_2\beta_2 + \ldots$$

$$(9) \qquad n = \mathfrak{b}_1\alpha_1 + \mathfrak{b}_2\alpha_2 + \ldots, \quad \delta = \mathfrak{b}_1\beta_1 + \mathfrak{b}_2\beta_2 + \ldots$$

The angles H may, from now on, be briefly designated as those of the first kind, the angles h as of the second kind.

If each term in (1) is differentiated, trigonometrical series of the same form appear which we will assume are also convergent. Let this be the case however often they are differentiated. Then we get

$$(10) \quad x_\lambda' = -\Sigma K_\lambda N \sin H, \quad y_\lambda' = \Sigma K_\lambda N \cos H, \quad z_\lambda' = \Sigma k_\lambda n \cos h,$$

or

$$(11) \quad x_\lambda' = -\Sigma G_\lambda \sin H, \quad y_\lambda' = \Sigma G_\lambda \cos H, \quad z_\lambda' = \Sigma g_\lambda \cos h,$$

if we put

$$K_\lambda N = G_\lambda, \qquad k_\lambda n = g_\lambda.$$

In the same way

$$(12) \qquad x'' = -\Sigma K_\lambda N^2 \cos H, \quad y'' = -\Sigma K_\lambda N^2 \sin H,$$
$$z'' = -\Sigma k_\lambda n^2 \sin h, \text{ etc.}$$

These series are to converge and to be unconditionally convergent. That is the sum of their absolute values,

$$\Sigma[K_\lambda N^p], \quad \Sigma(k_\lambda n^p)$$

are to converge for each integral exponent p.

From (1) it follows that

$$(13) \qquad r_\lambda^2 = x_\lambda^2 + y_\lambda^2 + z_\lambda^2 = \Sigma\Sigma K_\lambda K_\lambda' \cos(H - H')$$
$$+ \Sigma\Sigma k_\lambda k_\lambda' \cos(h - h'),$$

where the accent means that K_λ and K_λ', etc., belong to any two terms of (1) and the double summation signifies that the summation is to be carried out for the integral numbers in each term.

All the angles appearing in (13) are of the second kind, and it appears that r_λ may be developed in the form

(14) $$\Sigma k \cos h.$$

The same form (14) appears in the development of the squares of the distances of the planets from each other. One further assumption will be made and that is, that all positive and negative powers of the distances can be developed in trigonometrical series of the form (14).

The major axes of the orbits and the expressions (12), § 28, representing h, l, p, q can also be developed in trigonometrical series of the forms (1) and (14). In the narrower sense, the expression *stability* is now to be defined as follows:

1. *For the major axes, the constant term is decidedly larger than the sum of the coefficients of the periodic terms.*

2. *The sum of the coefficients in the trigonometrical series for the expressions* (12), § 28, *is small.*

The constant angles β_λ which are contained in l_λ replace $(3n-1)$ constants of integration. The coefficients a_λ in (4) and the coefficients K and k in (1) contain besides only $(3n-1)$ constants of integration. These are the secular values of the major axes of the previous paragraphs and the factors K and K' of § 31. Altogether there are $(6n-1)$ constants of integration or two less than is required in the complete solution with arbitrary choice of the system of coordinates.

The expressions (1) are obtained on the assumption that the origin falls at the sun's center. If the origin is put at the center of gravity of the $(n+1)$ bodies, (the sun and n planets) and this is convenient in the investigations that follow, the new coordinates are linear functions of the old and the forms of equations (1) remain unchanged.

In equations (1), let the sign Σ extend to all integral numbers $\mathfrak{a}$ or $\mathfrak{b}$. If the summation be extended to all the $(n+1)$ bodies, we may, to avoid confusion, use the symbol S.

Then

(15) $$Sm_\lambda x_\lambda = 0, \quad Sm_\lambda y_\lambda = 0, \quad Sm_\lambda z_\lambda = 0.$$

This gives, by (1), the relations

(16) $$Sm_\lambda K_\lambda(\mathfrak{a}_1, \mathfrak{a}_2, \ldots \mathfrak{a}_{3n-1}) = 0 \quad Sm_\lambda k_\lambda(\mathfrak{b}_1, \ldots \mathfrak{b}_{3n-1}) = 0,$$

which must hold for each system of numbers $\mathfrak{a}$ and $\mathfrak{b}$.

Further

(17) $$Sm_\lambda(x_\lambda y_\lambda' - y_\lambda x_\lambda') = \tfrac{1}{2} S\Sigma\Sigma m_\lambda K_\lambda K_\lambda'(N+N')\cos(H-H').$$

The first member is constant. The periodic terms in the second member must therefore vanish and thus a new series of relations arises between the coefficients K, while the second member is reduced to a constant term, making

(18) $$Sm_\lambda(x_\lambda y_\lambda' - y_\lambda x_\lambda') = S\Sigma m_\lambda K_\lambda^2 N = S\Sigma m_\lambda K_\lambda G_\lambda.$$

Further, we get

(19) $$Sm_\lambda(y_\lambda z_\lambda' - z_\lambda y_\lambda') = \tfrac{1}{2} S\Sigma\Sigma m_\lambda K_\lambda k_\lambda (N+n)\sin(H-h),$$

(20) $$Sm_\lambda(z_\lambda x_\lambda' - x_\lambda z_\lambda') = -\tfrac{1}{2} S\Sigma\Sigma m_\lambda K_\lambda k_\lambda (N+n)\cos(H-h).$$

The second members contain only terms of the first kind, hence no constant terms. They must, therefore, vanish. That is,

(21) $$Sm_\lambda(y_\lambda z_\lambda' - z_\lambda y_\lambda') = Sm_\lambda(z_\lambda x_\lambda' - x_\lambda z_\lambda') = 0,$$

as must be the case when the invariable plane is the plane of xy.

As shown before the expressions (1) contain $(6n-2)$ constants of integration, half of which are formed by the constant angles β contained in l and which do not appear in K, k and a. The other half which appear in K, k and a, can be represented by

(22) $$c_1,\ c_2,\ \ldots\ c_{3n-1}.$$

By Lagrange's theorem, (30), § 10, the expressions

(23) $$[a_i, a_j] = Sm_\lambda \left[\frac{\partial x_\lambda}{\partial a_i}\frac{\partial x_\lambda'}{\partial a_j} - \frac{\partial x_\lambda}{\partial a_j}\frac{\partial x_\lambda'}{\partial a_i} + \frac{\partial y_\lambda}{\partial a_i}\frac{\partial y_\lambda'}{\partial a_j} - \frac{\partial y_\lambda}{\partial a_j}\frac{\partial y_\lambda'}{\partial a_i} + \frac{\partial z_\lambda}{\partial a_i}\frac{\partial z_\lambda'}{\partial a_j} - \frac{\partial z_\lambda}{\partial a_j}\frac{\partial z_\lambda'}{\partial a_i} \right],$$

where a_i, a_j are any two of the above $(6n-2)$ constants of integration are constant. These may be developed by the method employed by Newcomb (*Théorie des perturbations de la lune qui sont dues à l'action des planètes. Journal des Mathématiques pures et appliquées, 1871*, and *Sur un problème de mécanique céleste. Comptes rendus, 1872.*

If we introduce the notation

$$(24)\qquad \{A, B\} = \frac{\partial A}{\partial \alpha_i}\frac{\partial B}{\partial \alpha_j} - \frac{\partial A}{\partial \alpha_j}\frac{\partial B}{\partial \alpha_i},$$

and substitute (1) and (11) in 23, it follows that

$$\begin{aligned}(25)\quad [\alpha_i, \alpha_j] = {} & S\Sigma\Sigma m_\lambda[\{K, G'\}\sin(H-H') + \{k, g'\}\sin(h-h')] \\ & - S\Sigma\Sigma m_\lambda[G'\{K, H'\}\cos(H-H') + g'\{k, h'\}\cos(h-h')] \\ & + S\Sigma\Sigma m_\lambda[K\{H, G'\}\cos(H-H') + k\{h, g'\}\cos(h-h')] \\ & + S\Sigma\Sigma m_\lambda[KG'\{H, H'\}\sin(H-H') \\ & \qquad + kg'\{h, h'\}\sin(h-h')].\end{aligned}$$

If we put

$$(a)\qquad \alpha_i = \beta_i, \quad \alpha_j = \beta_j,$$

only the last term in (25) remains, since K, k and α are independent of β. By (24), we have here

$$\{H, H'\} = \mathfrak{a}_i\mathfrak{a}_j' - \mathfrak{a}_j\mathfrak{a}_i', \quad \{h, h'\} = \mathfrak{b}_i\mathfrak{b}_j' - \mathfrak{b}_j\mathfrak{b}_i',$$

and therefore

$$\begin{aligned}(26)\qquad [\beta_i, \beta_j] = S\Sigma\Sigma m_\lambda[& (\mathfrak{a}_i\mathfrak{a}_j' - \mathfrak{a}_j\mathfrak{a}_i')KG'\sin(H-H') \\ & + (\mathfrak{b}_i\mathfrak{b}_j' - \mathfrak{b}_j\mathfrak{b}_i')kg'\sin(h-h')].\end{aligned}$$

This is to be constant. All periodic terms destroy one another. There is no constant term. Hence

$$(27)\qquad [\beta_i, \beta_j] = 0.$$

$$(b)\qquad \alpha_i = \beta_i, \quad \alpha_j = c_j.$$

We have .

$$\{H, G'\} = \mathfrak{a}_i\frac{\partial G'}{\partial c_j}, \quad \{h, g'\} = \mathfrak{b}_i\frac{\partial g'}{\partial c_j},$$

$$\{K, H'\} = -\mathfrak{a}_i'\frac{\partial K}{\partial c_j}, \quad \{k, h'\} = -\mathfrak{b}_i'\frac{\partial k}{\partial c_j},$$

$$\{H, H'\} = t\left(\mathfrak{a}_i\frac{\partial N'}{\partial c_j} - \mathfrak{a}_i'\frac{\partial N}{\partial c_j}\right), \quad \{h, h'\} = t\left(\mathfrak{b}_i\frac{\partial n'}{\partial c_j} - \mathfrak{b}_i'\frac{\partial n}{\partial c_j}\right).$$

The remaining combinations $\{A, B\}$ in (25) vanish, hence

$$\begin{aligned}(28)\quad [\beta_i, c_j] = S\Sigma\Sigma m_\lambda\Bigg[& \left(\mathfrak{a}_i K\frac{\partial G'}{\partial c_j} + \mathfrak{a}_i' G'\frac{\partial K}{\partial c_j}\right)\cos(H-H') \\ & + \left(\mathfrak{b}_i k\frac{\partial g'}{\partial c_j} + \mathfrak{b}_i' g'\frac{\partial k}{\partial c_i}\right)\cos(h-h')\Bigg]\end{aligned}$$

$$+tS\Sigma\Sigma m_\lambda\Big[\Big(\mathfrak{a}_i\frac{\partial N'}{\partial c_j}-\mathfrak{a}_i'\frac{\partial N}{\partial c_j}\Big)KG'\sin(H-H')$$
$$+\Big(\mathfrak{b}_i\frac{\partial n'}{\partial c_j}-\mathfrak{b}_i'\frac{\partial n}{\partial c_j}\Big)kg'\sin(h-h')\Big].$$

Since $[\beta_i, c_j]$ is constant, the terms multiplied by t must vanish and the above must reduce to the constant term. Hence

$$(29)\qquad [\beta_i, c_j]=S\Sigma m_\lambda\Big(\mathfrak{a}_i\frac{\partial(KG)}{\partial c_j}+\mathfrak{b}_i\frac{\partial(kg)}{\partial c_j}\Big)$$
$$=\frac{\partial}{\partial c_j}(S\Sigma m_\lambda(\mathfrak{a}_iK^2N+\mathfrak{b}_ik^2n)).$$

$$(c)\qquad a_i=c_i,\quad a_j=c_j.$$

Here we have

$$(30)\qquad [c_i, c_j]=S\Sigma\Sigma m_\lambda\Big[\Big(\frac{\partial K}{\partial c_i}\frac{\partial G'}{\partial c_j}-\frac{\partial K}{\partial c_j}\frac{\partial G'}{\partial c_i}\Big)\sin(H-H')$$
$$+\Big(\frac{\partial k}{\partial c_i}\frac{\partial g'}{\partial c_j}-\frac{\partial k}{\partial c_j}\frac{\partial g'}{\partial c_j}\Big)\sin(h-h')\Big]$$
$$+tS\Sigma\Sigma m_\lambda[(K\{N, G'\}-G'\{K, N\})\cos(H-H')$$
$$+(k\{n, g'\}-g'\{k, n'\})\cos(h-h')]$$
$$+t^2S\Sigma\Sigma m_\lambda[KG'\{N, N'\}\sin(H-H')$$
$$+kg'\{n, n'\}\sin(h-h')].$$

The periodic terms and all terms multiplied by t and t^2 must destroy one another. A really constant term does not occur in (30). Hence

$$(31)\qquad [c_i, c_j]=0.$$

The vanishing factor in the term proportional to t is

$$0=S\Sigma m_\lambda\Big[\Big(\frac{\partial N}{\partial c_i}\frac{\partial(KG)}{\partial c_j}-\frac{\partial N}{\partial c_j}\frac{\partial(KG)}{\partial c_i}\Big)\cos(H-H')$$
$$+\Big(\frac{\partial n}{\partial c_i}\frac{\partial(kg)}{\partial c_j}-\frac{\partial n}{\partial c_j}\frac{\partial(kg)}{\partial c_i}\Big)\cos(h-h')\Big)\Big].$$

When the values of N and n given in (8) and (9) and equation (29) are used, the above equation takes the form

$$(32)\qquad 0=\frac{\partial a_1}{\partial c_i}[\beta_1, c_j]+\frac{\partial a_2}{\partial c_i}[\beta_2, c_j]+\ldots+\frac{\partial a_{3n-1}}{\partial c_i}[\beta_{3n-1}, c_j]$$
$$-\frac{\partial a_1}{\partial c_j}[\beta_1, c_i]-\ldots\qquad-\frac{\partial a_{3n-1}}{\partial c_j}[\beta_{3n-1}, c_i].$$

Equation (32) can be used for the deduction of a noteworthy theorem. The quantities c are not more closely determined than that they appear in the constants K, k, α. If desired the $(3n-1)$ quantities α may be considered as the c's. But any $(3n-1)$ independent functions of K, k, α may be taken for the c's and it is convenient, in order to make (29) as simple as possible, to put

$$c_i = S\Sigma m_\lambda(\mathfrak{a}_i K^2 N + \mathfrak{b}_i k^2 N). \tag{33}$$

In this case

$$[\beta_i, c_i] = 1, \tag{34}$$

while all the other Lagrange combinations between β and c vanish.

When the constants c_i have been determined by (33), it remains to introduce them into K, k, α, that is to determine these coefficients as functions of c_i. We will not undertake this determination, but only give the theorem referred to above. With the aid of (34), (32) becomes

$$\frac{\partial \alpha_j}{\partial c_i} = \frac{\partial \alpha_i}{\partial c_j}. \tag{35}$$

This equation shows that the quantities α, when expressed in terms of the c's, are the partial derivatives of a function θ with respect to the c's. That is,

$$\alpha_i = \frac{\partial \theta}{\partial c_i}. \tag{36}$$

The function θ can now be more closely determined, but not in the desired form, for it will not be expressed in terms of the c's but in α, K and k. It is evident that the α's must be homogeneous functions of the c's. Let p be the degree of these functions. The quantities K and k are also homogenous function of the c's. Let their degree be q. Then the coefficients x', y', z' are of the degree $p+q$ and of x'', y'', z'', $2p+q$. On the other hand, the coefficients of $x'' = \frac{\partial V}{\partial x}$, etc., are of the degree $q - 3q - = -2q$. Hence

$$2p + q = -2q, \quad 2p = -3q.$$

Further, it follows from (33), when the degree of the terms in the first and second members are compared, that

$$1 = 2q + p,$$

and therefore,

(37) $$p = -3, \quad q = 2.$$

Since the a's are homogenous functions of the c's of the degree -3, the degree of θ must be -2. Hence

$$\frac{\partial \theta}{\partial c_1} c_1 + \frac{\partial \theta}{\partial c_2} c_2 + \ldots + \frac{\partial \theta}{\partial c_{3n-1}} c_{3n-1} = -2\theta,$$

and, by (36), (33), (8) and (9)

(38) $$-2\theta = S \Sigma m_\lambda (K^2 N^2 + k^2 n^2).$$

The second member may be brought into a noteworthy form. The kinetic energy T is, by (1),

(39) $$T = \tfrac{1}{2} S \Sigma m_\lambda (KNK'N' \cos(H - H') + knk'n' \cos(h - h')).$$

If the constant term of the second member (Clausius' *Viriel*) be represented by V, then

(40) $$V = -\theta,$$

and

(41) $$a_i = -\frac{\partial V}{\partial c_i}.$$

Up to this time the invariable plane has been taken as the plane of xy. In order to give the most general representation to the coordinates, without fixing the xy plane in any way, the previous coordinates x, y, z will be replaced by new ones ξ, η, ζ by the transformation formulas

$$\xi = a_1 x + a_2 y + a_3 z,$$
$$\eta = b_1 x + b_2 y + b_3 z,$$
$$\zeta = c_1 x + c_2 y + c_3 z.$$

The transformation coefficients can be expressed in terms of the three angles f, φ, δ, as follows:

$$a_1 = \cos f \sin \delta,$$
$$b_1 = -\sin f \cos \varphi - \cos f \sin \varphi \cos \delta,$$
$$c_1 = \sin f \sin \varphi - \cos f \cos \varphi \cos \delta,$$

$$
\begin{aligned}
a_2 &= \sin f \sin \delta, \\
b_2 &= \cos f \cos \varphi - \sin f \sin \varphi \cos \delta, \\
c_2 &= -\cos f \sin \varphi - \sin f \cos \varphi \cos \delta, \\
a_3 &= \cos \delta, \\
b_3 &= \sin \varphi \sin \delta, \\
c_3 &= \cos \varphi \sin \delta.
\end{aligned}
$$

If, for brevity, we put

$$
\begin{aligned}
x \cos f + y \sin f &= (x) = \Sigma K \cos (H - f), \\
-x \sin f + y \cos f &= (y) = \Sigma K \sin (H - f),
\end{aligned}
$$

we get

$$
(42) \qquad \begin{cases} \xi = (x) \sin \delta + z \cos \delta, \\ \eta = -(x) \sin \varphi \cos \delta + (y) \cos \varphi + z \sin \varphi \sin \delta, \\ \zeta = -(x) \cos \varphi \cos \delta - (y) \sin \varphi + z \cos \varphi \sin \delta. \end{cases}
$$

It appears that the angle f occurs only in the combination $(H-f)$. In fact f is a surplus constant, for it is only necessary in formulas (1) to lessen all the β's by f to change H into $(H-f)$, while the angles h remain unchanged. It is therefore not necessary to retain the angle f, and formulas (42) can be directly used, and x and y may be written for (x) and (y). The previous $[a_i, a_j]$ remain unchanged by the transformation (42). Further, it follows that

$$
(43) \qquad [\delta, c_i] = [\delta, \beta_i] = [\varphi, c_i] = [\varphi, \beta_i] = 0,
$$

and, finally,

$$
(44) \qquad [\varphi, \delta] = S m_\lambda (x y' - y x') \sin \delta = S \Sigma m_\lambda N K^2 \sin \delta.
$$

If the motion is referred to a stationary system of coordinates, the coordinates of the center of gravity must be added. These are, by (9), § 6, linear functions of the time. It appears, therefore, that

$$
(45) \qquad [a, a_1] = [b, b_1] = [c, c_1] = -\frac{1}{\Sigma m}.
$$

All of Lagrange's combinations have now been formed. The system of $(6n+6)$ constants of integration can easily be changed into a cannonical one, but we will not occupy ourselves with it.

The fundamental idea in the present problem is, therefore, the following:

Take the system

$$(46)\qquad \frac{dp_i}{dt}=\frac{\partial H}{\partial q_i},\quad \frac{dq_i}{dt}=-\frac{\partial H}{\partial p_i},\quad (i=1,2,\ldots n)$$

in which H does not explicitly contain the time. Change p_i, q_i by a canonical transformation into the new variables P_i, Q_i and the new system is

$$(47)\qquad \frac{dP_i}{dt}=\frac{\partial H}{\partial Q_i},\quad \frac{dQ_i}{dt}=-\frac{\partial H}{\partial P_i},\quad (i=1,\ldots n).$$

H is to be expressed as a function of the new variables. If we can now succeed in selecting such a canonical transformation, that H contains only half of the new variables, say P_i the problem is solved. For then $\dfrac{\partial H}{\partial Q_i}=0$, and, therefore, $P_i=$ a constant. At the same time the $\dfrac{dQ_i}{dt}$ would be constant and the Q_i's would be linear functions of the time. In our case the present P_i is the former c_i and the Q_i the l_i.

These are the results to which we are led by the assumption that form (1) of the solutions obtained by induction is a mathematical law. If they do not suffice actually to give a solution, they afford a new view of Celestial Mechanics.

A few brief remarks follow.

The coordinates have been represented as trigonometrical functions of the $(3n-1)$ angles,

$$l_1, l_2 \ldots l_{3n-1}.$$

These can be replaced by any $(3n-1)$ others as

$$l_1', l_2', \ldots l_{3n-1},$$

which are formed from the preceding by linear substitutions, with integral coefficients. For example

$$l_\lambda' = a_{1,\lambda}\, l_1 + a_{2,\lambda}\, l_2 + \ldots + a_{3n-1,\lambda}\, l_{3n-1},$$

provided the substitution determinant $=1$. For then the solutions of (48) become

$$(49)\qquad l_\lambda = b_{1,\lambda} l_1' + b_{2,\lambda} l_2' + \ldots + b_{3n-1,\lambda}\, l_{3n-1}',$$

containing also integral coefficients. If H is a linear function of the l's with integral coefficients, then, by (49), it is changed into a similar function H' of the l''s and *vice versa.*

If the problem of $(n+1)$ bodies is limited to the distances only, angles of the second kind, h, appear and for these the sum of the integral numbers vanishes. Hence, these angles can also be written

$$\begin{aligned} h &= b_1 l_1 + b_2 l_2 + \ldots + b_{3n-1} l_{3n-1} \\ &= b_2(l_2 - l_1) + b_3(l_3 - l_1) + \ldots + b_{3n-1}(l_{3n-1} - l_1), \end{aligned}$$

so that they depend only on the $(3n-2)$ angles

$$(l_2 - l_1), (l_3 - l_1) \ldots (l_{3n-1} - l_1).$$

Any angle of the first kind can be changed into

$$\begin{aligned} H &= \mathfrak{a}_1 l_1 + \mathfrak{a}_2 l_2 + \ldots + \mathfrak{a}_{3n-1} l_{3n-1} \\ &= l_1 + \mathfrak{a}_2(l_2 - l_1) + \ldots + \mathfrak{a}_{3n-1}(l_{3n-1} - l_1). \end{aligned}$$

Therefore:

If $(n+1)$ bodies are moving about their center of gravity, their distances will be trigonometrical functions of $(3n-2)$ angles, all of which are linear functions of the time. In order to get the coordinates, another angle of the same form must be added.

For $n = 1$, this result is completely verified. The distance of the planet from the sun is a trigonometrical function of the mean anomaly. In order to gêt the coordinates x and y (z here $= 0$), another angle is needed which is here constant and is equal to the longitude of perihelion.

For $n = 2$, Lindstedt has formally deduced the same result, leaving questions of convergence aside, from Lagrange's differential equations for three bodies, (38) and (45), § 7. (See his *Sur la détermination des distances dans le problème des trois corps. Annales de l'école normale,* 1884). If the coefficients α of the time t in

$$l_\lambda = \alpha_\lambda t + \beta_\lambda$$

are incommensurable, the original coordinates never reappear. If they have a greatest common measure C, they are integral multiples of c, and so are N and n. The coordinates can then be developed in trigonometrical functions of a single angle ct,

and after the time $T=\frac{2\pi}{c}$, the system returns to its original position, so that the motion is purely periodic, exactly as with two bodies.

The special cases of the problem of three bodies have been given in § 8. Liouville has shown (*Mémoire sur un cas particulier du problème des trois corps, Journal des Mathématiques pure et applique*, 1842, page 110 and 1856 page 248) that the special case in which the three bodies remain constantly in a straight line and describe similar ellipses in space, is unstable in so far as a very small change in the initial position, such as would follow from the smallest thrust or force from without, would, in time cause the three lines joining the bodies to become fully separated. From what precedes this would be expected. For if, in a special case, the α's are commensurable, they cease to be so by a very small departure from the original position. The ratios of the periods are, then, not expressible in whole numbers and therefore the special condition, that the three points shall lie in a straight line, must finally be lost.

If the α's are incommensurable, the motion is not purely periodic. Every angle l has, to be sure, its period, but these periods do not agree, and consequently the motions presented to the eye are very complicated, such for example, as are exhibited by the paths of the components of many of the multiple stars.

42. Brief Historical Review of the Theories of Perturbations.

The history of the theories of perturbations can be carried back to Newton, who, in the third book of his *Principia* explains the chief of irregularities of the moon's motions. The next advance was in 1747 when Clairaut and d'Alembert presented to the French Academy of Sciences two memoirs on the problem of three bodies, in which they gave methods of integrating the differential equations when the central force is the controlling one. They developed the fundamental parts of the theory of absolute perturbations and Clairaut soon had an opportunity

to give a brilliant test to his theory. Halley's comet had been seen near perihelion in 1531,.1607 and 1682, and it was settled that its period was about 75 years. Its return was expected, therefore, at the latest, about the end of 1758. Clairaut computed its perturbations due to the known planets and informed the French Academy that its return would be delayed until about the middle of April 1759. This announcement was made on November 14, 1758. He added that the small terms neglected in his calculation of the retardation, which he put at 690 days, might make a difference of a month. In fact, the re-appearance was on March 12, 1759, within the limits which he had set, and he would have been more exact had he known Saturn's mass more closely.

The founder of the theory depending on the method of the variation of constants was Leonhard Euler. In three memoirs, which were crowned by the French Academy in 1748, 1752 and 1756, he developed by regular steps the method of the variations of constants. His formulas were not entirely rigorous for among the six constants there were some which he did not vary. Yet for the case of two planets, he succeeded in showing the existence of secular variations of eccentricities, inclinations, perihelia, and nodes. Unfortunately, numerous errors crept into the numerical developments, completely vitiating the results and possibly causing Euler to devote himself to other investigations. Besides the foundation of the general theory, he gave, in his papers, the fundamental propositions for the development of the perturbing function in a trigonometrical series whose arguments are the mean anomalies.

Euler's ideas, which laid the foundations, were developed to their limits by Lagrange, whose services to astronomy alone are enough to give him an indestructible memorial. His first work appeared in the *Mélanges de la Société de Turin, Tome III*, 1766. He here treated the eccentricity and longitnde of perihelion, in the expressions for the radius vector, as variable, and in this sense differentiated totally with respect to the time. The part of the derivatives which depends on the variation of these two constants, he equated to zero, differentiated again, then intro-

duced the perturbing function, and thus got two equations for determining the derivatives of these two elements. This process is undoubtedly inexact, because it disregards the variations of the major axis and of the epoch of perihelion. Yet he formed, in the case of two planets, the correct final equation for for determining the secular periods, which Euler had given erroneously. He also obtained correct formulas for the inclinations and nodes, so that his numerical results in an application to Jupiter and Saturn were nearly correct. On the other hand in the determination of the mean longitudes of the two planets, two terms proportional to the squares of the times were included which are entirely wrong and which were due to the inaccuracies which the great mathematician permitted in this first attempt in this uncultivated field of science.

In 1773, Laplace offered to the French Academy of Sciences his first work on the theory of the planetary system, and in 1776 this was printed in the *Mémoires des savants étrangers.* He determined with entire accuracy the formulas for the derivatives of the elements, but these did not possess their present elegant form.

While he was carrying on the numerical computations for Jupiter and Saturn, he found to his great surprise that the secular terms in the values of the derivatives of the major axes, mutually vanished. He succeeded in proving that this phenomenon was not a play of chance, but founded in the formulas when their development was carried out to terms of the second degree in the eccentricities and inclinations.

The theory of secular variations of the elements originated fundamentally in the circumstance that the usual process of integration by series brought to light terms proportional to the time. Every device was employed by Euler, Lagrange, and Laplace to get rid of these terms. Any one who studies their works, must feel that aside from mathematical considerations, a sort of metaphysical idea directed these earliest explorations which bridged the gap between entire ignorance and complete clearness. It soon appeared that the theory of absolute perturbations was not the best, and the variation of constants grad-

ually displaced it, though curiously enough, not in its purity, but wonderfully mixed with the first theory. It was seen that the secular terms must eventually cause great changes in the elements, but instead of determining these directly, an indirect course was taken. This appears plainly in Laplace (*Mécanique céleste*, first volume), where, in the expressions for the absolute perturbations, the terms proportional to the time are forcibly removed, and then after introducing the complete theory of variation of constants, the natural path of integration is taken, retaining only the secular terms of the perturbing function.

Another advance was made by Lagrange in 1774, in a paper which appeared in the *Annales* of the French Academy, where he introduced the elements h and l in place of the eccentricities and perihelia and p and q in place of the inclinations and nodes. The differential equations became linear by this transformation. In 1780, in a paper on the perturbations of comets, which was crowned by the Berlin Academy and published in their memoirs in 1782, he treated the expressions for elliptic coordinates for the first time in their most general form, that is, not as functions of the time alone, but also of the six constants of integration. Here for the first time the theory of the variation of constants was demonstrated with entire rigor and purity, and the formulas for its use simplified. When Lagrange applied his general theory to the expression for the major axes, he recognized at once that in $\frac{da}{dt}$ only periodic terms appear, and thus,—with the stroke of a pen, says Jacobi,—the proposition was proven, that the influence of the perturbations on the major axes is proportional to the disturbing masses only, and is expressible by periodic terms so long as only the first powers of the disturbing masses are considered.

In 1809, Poisson in his paper *Sur les inégalités séculaires des moyens mouvements des planétes* (*Journal de l'école polyt. I*) showed that this proposition still held if the second powers of the masses were included. He obtained this result by a peculiar use of the theorem of kinetic energy. Laplace gave in *Mécanique céleste* another and more complete demonstration which has

been followed here. Lagrange also tried to provide a proof, but, unfortunately his process suffered from an error in sign, which Serret has pointed out.

Endeavors have been made to extend this demonstration to the third powers of the disturbing masses. Mathieu, (*Mémoires sur les inegalités séculaires des grands axes des orbites des planètes, Crelle's Journal*, 1875), carried out investigations from which he concluded that the secular terms fail even when the third powers are included. Later Tisserand, Harétu, Gasparis, Gyldén, and others have gone over the same ground with a result sometimes favorable, sometimes unfavorable, so that the question is not finally settled.

After the formulas for the derivatives of the elements had been developed, their notable simplicity in the case of the axes led to a search for equally simple expressions for the other elements. Laplace, by skillful changes succeeded in producing the fundamental formulas, as given in (8), § 28. The discovery was published in 1808. By a remarkable coincidence, Lagrange published the same formulas at the same session of the *Bureau des longitudes.* He had reached them in an entirely different way, namely, by the introduction of the expressions (a_λ, a_μ) and he at the same time discovered the peculiar origin of these important formulas. With these the theory won a wonderful simplicity and elegance, as compared with the earlier forms, and astronomers were enabled to replace the derivatives of the perturbing function with respect to coordinates, by those with respect to the elements.

The denominator which appears in the periodic terms on integration is a very notable phenomenon, to which reference has already been made. Laplace deserved the credit of having first called attention to it, and of having investigated the remarkable increase of an otherwise small periodic term, caused by this denominator when small. He applied the theory to Jupiter and Saturn and thus explained a very enigmatical phenomenon. Halley had compared ancient observations with those of the middle ages, and had drawn the conclusion that the motion of Saturn about the sun had been retarded, that of Jupi-

ter accelerated. On the other hand, Lambert had found that in modern times the opposite is the case. Laplace saw that the explanation of this apparent contradiction of the law of the invariability of motions was to be found in a term of the perturbing function, neglected up to that time because of its minuteness, and he fixed the period of this perturbation at 930 years. These purturbations of long periods, as they are called, are the source of many others, and they especially affect the secular elements.

If the periods are yet more nearly commensurable, the period continually becomes longer and the coefficient continually increases. Finally a phenomenon occurs which actually takes place in another form in our system. The two planets affect each other in such a way that their periods become exactly commensurable, and a close bond is formed between them, which continues forever unbroken. It lacked but little, as Laplace showed, of occurring in the case of Jupiter and Saturn. To assure it, it would only be necesssary to decrease Saturn's mean distance by $\frac{1}{520}$ and to increase Jupiter's by $\frac{1}{1300}$.

After the theory had been thoroughly grounded, and our knowledge had passed from the darkness of its origin into the stadium of perfect clearness, the effort was to bring the analytical and numerical developments to the highest degree of sharpness and exactness. When the perturbing function is developed to higher powers of the eccentricities and inclinations, the labor involved rapidly increases. This induced the most eminent mathematicians to occupy themselves in improving the solution. Cauchy published in the *Comptes rendus* a series of many articles in which he advanced entirely new ideas, and used his celebrated theory of residuals. He expressed the coordinates in terms of the eccentric anomalies and developed the perturbing function in trigonometrical functions of them. It was then easy, by Bessel's functions, to pass from the eccentric to the mean anomalies. His formulas are of especial elegance and simplicity when the term, the coefficients of which are to be determined, is of high order. His labors in this field do not appear to have found much appreciation among other astrono-

mers although he showed, in a controversy with Leverrier, that his results are of practical value.

When it is only the question of the numerical value of a coefficient, a double integration will answer, as has been shown. It it noteworthy that Liouville (*Note sur le cacul des inégalités periodiques du mouvement des planètes. Journal des Mathématiques pure et appliquèes,* 1836, page 197), offered a process by which this double integral can be reduced to a simple one with a high degree of exactness. Leverrier, at a later time made a searching investigation with the numerical and also analytical development of the perturbing function, and he reached a remarkable degree of accuracy. In his *Recherches astronomiques,* he calculated all the coefficients up to and including the seventh degree, and performed a labor which only he can judge who has once undertaken such computations. Besides the astronomers and mathematicians already named, Bessel, Lubbock, Encke, Hansen, Gyldén, Newcomb and others have worked at the development of the perturbing function. Hansen, who prefers his own method, believes that, when the eccentricity and inclination of the disturbed planet are relatively great, the most convenient process is to develop the perturbing function with respect to the eccentric anomaly of this planet and the mean anomaly of the disturbing planet.

Laplace and Lagrange developed the secular values of the elements with limitation to the second degree of the eccentricities and inclinations, but it was left to Leverrier, so far as I know, to undertake the estimate of the influence of the neglected terms. His investigations were continued by Lehmann, who died before the computation of the numerical work. Leverrier and Lehmann concluded that the influence of the neglected terms is greater than had been assumed, and that possibly great improvements in the secular periods can be made as they were determined by the roots of the secular equations.

The theories of perturbations afford much opportunity for changes and transformations, and this is employed to attack the problem in new ways. Hansen, for instance, introduced a moving system of coordinates in order by its use to refer the mo-

tions of the planet about the sun to one plane. He imagined each point of the orbit connected with the sun and thus formed a very flat cone with the sun at the apex. On this cone rolls a plane without slipping, so that at each instant a point in the plane coincides with a point in the orbit. In this manner the orbit is described in one plane. Two differential equations of the second order determine this plane, and three of the first order determine its rotation. In addition, he uses two symbols t and τ, for the time, the one direct in the elliptic coordinates, the other indirect in the perturbations, and he develops certain views in which he varies the latter term τ. Since his investigations relate to the moon, minor planets and comets, for which they are of great importance, they will not be noticed further, though they are not without mathematical interest.

The same is true of Delaunay's investigations. His process is to fix the attention on a single term, and especially the most important of the developed perturbing function, and to integrate the differential equations thus obtained. This is possible with mathematical rigor. With the aid of these integrals the differential equations are changed and another prominent term is considered. This is done repeatedly until the remainder is so small that it may be expeditiously treated by the usual methods.

More recently Glydén has attacked the planetary problem in a different way. In order to get the special perturbations, he divides the orbit into parts and develops the purturbations for each part separately, then introduces elliptic functions into the elliptic elements. The latter he favors especially because, with their help, the integrations converge more rapidly. He has also introduces a peculiar conception, that of the intermediate orbit, by which he means an orbit from which the planet never widely departs. In so far there is nothing new in this conception; it is already introduced in § 33. It is to be noted, however, that he does not get his intermediate orbit from Kepler's ellipse but reaches it directly. The reader can find a part of his principles in *Die intermediäre Bahn des Mondes* (*Acta Math. VIII*), and in a paper by Andoyer,—*Contribution à la théorie des orbites intermédiares*, (*Annales de la faculté des sciences de Toulouse*,

1887, I). He has lately undertaken studies on the convergence of the infinite series used in astronomy, which have, however so far led to no satisfactory conclusion.

One of the chief aims of mathematical effort of the present time consists in the investigation of the properties of functions defined by differential equations. A glance at the immense number of investigations, which the theory of linear differential equations alone has called out in very recent times, and is daily calling out, will give a clear idea of the great magnitude of the problems here involved, and will afford a conception of the difficulties which lie in the way of a complete solution of the problem of three bodies in its most general form. The question here is, certainly, concerning real variables only, and this circumstance simplifies the problem greatly. What interests the astronomers is not the investigation of all the properties of the analytical expressions, in finding their nodes, acnodes, cusps, etc., but the proof that such points do not exist when the time is made the independent variable and extends over a real path uninterruptedly from $-\infty$ to $+\infty$. The question, therefore, with this limitation to one branch, or rather to one line of a function, relates to the development of the coordinates in converging series, and these must converge without fail. It is easy to see that this can be done without embracing the great reverse problem of differential equations, which affords us already so many noble results. There must be a direct road, in the above sense, to the solution of the problem of many bodies, a path which will lead to correct answers to the questions which, in spite of all investigations, still remain unanswered.

Purely theoretical discussions are not sufficient for the practical aims of astronomy. The numerical computations, also, which the author of this book has not undertaken, but has forced entirely in the background in order to preserve the analytical character of the work, require great skill and greater patience. The calculations take on, in fact, almost unmanageable dimensions when, as Leverrier has done, in the *Recherches astronomiques* (*Annales de l'Observatoire de Paris, Mémoires*), all terms are taken into account which change the geocentric place

of the planet by the tenth of a second. The comparison of the computed places with the observed afford the best touchstone for the correctness of the theory, and it has very often led to the discovery of neglected but influential terms. It also affords a means of continuously correcting the masses and elements of the planets.

When Leverrier had completed his comparison of the computed and observed places of the planets, the result was found to be very satisfactory. The differences fell usually within the limits of error of observation and in only a few small matters was further discussion necessary. For instance, in the case of Mercury's perihelion he found that observation gave a somewhat greater movement than computation. He concluded that between Mercury and the Sun there was one planet, or perhaps many, which had so far escaped observation, and which would cause the acceleration of Mercury's perihelion. This intra-mercurial planet, to which the name Vulcan had already been given, has not yet been observed beyond a possibility of doubt, though it has been actively sought. Small, dark points have been seen, several times, to cross the sun's disk but it appears probable that they were either sun-spots or an error was involved in the observation.

The planetary tables constructed on Leverrier's formulas are to-day generally recognized, except that for Uranus and Neptune Newcomb's are better.

Of course, Newton's law of gravitation must submit to the test of observation, out of which it was originally created. Whether gravitation is a true *prima causa* or whether it results from other forces, either impulses, for instance, or electric, attraction, cannot now be decided. Possibly Newton's law is only approximate, though remarkably close. Possibly other forces are at work in celestial space, such as the resistance of the ether which, according to physicists at present, not only occupies stellar space but interpenetrates matter. Encke adopted the idea of such a resistance as a result of his investigations of the orbit of the comet named after him.

Certainly the law of gravitation is, as already said, very nearly fulfilled, so nearly that mathematicians must assume it rigorously true and investigate its consequences.

How closely it agrees with observation was shown in the most brilliant way by the discovery of Neptune, forty or fifty years ago. Sir W. Herschel discovered Uranus in 1781. It was then continuously observed and its elements computed. The necessary calculations were made by Bouvard according to Laplace's formulas. It appeared, however, that in the course of time the new planet departed in a regular manner from its computed place, and that the amount of departure increased with the time. It also appeared that the planet had been observed several times as a fixed star and that it was impossible satisfactorily to represent these old observations. Bouvard himself had concluded as early as 1821 (*Tables astronomiques, les mouvements d'Uranus annoncent l'existence d'une planète perturbatrice extérieure*), that a planet exists outside of Uranus. It was beyond Uranus, because otherwise it would appreciably affect Saturn. It gradually became a familiar idea that the mass and elements of the new planet could be computed from the observed purturbations, making the reverse of the ordinary problem. If the smallness of the perturbations be considered, they amounted to only a few minutes, and the complicated analytical form in which the elements appear in perturbing function, it will be evident that the problem was one of great refinement. Bessel interested himself actively in it and his letters show that he was about to take it up when death put an end to his activity. It was solved, almost simultaneously, by two other astronomers, Leverrier and Adams. The first published his preliminary investigations in the *Comptes rendus* and explained his research in the greatest detail in *Recherches sur les mouvements de la planète Herschel*, (*Connaissance des temps pour* 1849) after the new planet had been found in 1846 by Galle within a degree of the place he had predicted. The study of this work affords great pleasure because in it are combined great discretion, painful accuracy in analytical and numerical developments, and especially clearness in the development of the line of thought. The first

question was whether any terms of the development had been neglected which could cause an appreciable error. In fact, there were several suspected, but they were shown to be unimportant. Then the inquiry was, could the errors be eliminated by small changes in the elements selected and a negative result was obtained. These preparatory studies put beyond doubt the perturbing influence of an unknown cause, and, after the rejection of other possibilities, the perturbations were traced to a planet more distant than Uranus from the Sun. The major axis, the epoch of perihelion, eccentricity, longitude of perihelion and the mass of the new planet were introduced as the five unknown quantities, while the inclination with reference to Uranus was neglected because of the small perturbations in latitude. Each observation gave an equation of condition between five unknowns, and since the number of observations was very large, they were divided into groups. The major axis appeared in a very complicated form, hence Leverrier assumed, using Bode's law as a single foothold, that the new planet was twice as far from the sun as Uranus. The four other unknowns were then determined. Finally, he found that a decrease in the adopted value of the major axis decreased the unavoidable errors. He then determined the amount of the decrease and deduced the final value of the elements.

Undoubtedly Leverrier would have reached much closer results if he had not been misled by Bode's law and adopted at the beginning, a value much too great for the major axis of the unknown planet. It is evident that an increase in distance required an increase in the mass of the planet and as the unknown planet was nearly in conjunction with Uranus the error in distance had its greatest effect. It is due to this that the mass obtained by him was nearly double the real value and hence all the elements were entirely wrong except the mean longitude, which, in this case, was the most important, because the one fixing the place in the sky.

The same was true of the elements obtained by Adams, who published his investigations in *An Explanation of the Observed Irregularities in the motion of Uranus*, (*Memoirs of*

the Royal Society of London, 1847). As he found a still greater major axis his mass was even larger than Leverrier's. Since his results were communicated to Airy some months before Leverrier's investigations appeared, the planet might have been discovered earlier, had not Airy, who had some doubts as to the correctness of Adams's results, decided on making a catalogue of the stars in that part of the sky.

It is one of the best possible proofs of Newton's law of gravitation that it led to a discovery which might otherwise have been delayed many years.

43. Notes on the Tables.

The unit of distance is half the major axis of the earth's orbit, or the mean distance of the earth from the sun. This is not the semi-axis which we have called the secular value, but the one which follows from the observed periodic time and Kepler's third law. The secular value is obtained by applying the correction given in § 32. Of the masses, those of Mercury and Venus are the most uncertain. As these planets are not known to have satellites, their masses can be obtained only by the perturbations they produce on comets which pass near them and on the earth. It is quite possible that the mass of Mercury, especially, requires a very appreciable correction.

The secular variations of the longitude of perihelion, of the ascending node, and of the inclination are not referred to a fixed plane, but to the moving ecliptic on which the vernal equinox is selected as the initial point for longitudes. The vernal equinox retreats yearly on account of precession, a distance of $50''.2113$ but is free from nutation. If these variations are referred to the ecliptic of 1850, then during the current thousand years all perihelia advance, except that of Venus, while all the nodes retrograde, except those of Jupiter and Uranus.

The fifth table gives the roots of the equations (21) and (42) § 31; the sidereal year is taken as the unit of time. The amounts by which G and Γ annually increase and decrease is therefore, measured in seconds. The absolute maximum of the

increase is γ_4. If T is the time taken for the proper angle $\Gamma_4 = \gamma_4 t + \delta_4'$, to increase 360°, it follows that

$$T\gamma_4 = 1296000'',$$
$$T = 49937,$$

or in round numbers, 50,000 years. The longest period is about 2,000,000 years. From this the reader can see how slowly the secular variations of the elements change. The numerical coefficients K and K', which Leverrier computed anew in his *Recherches astronomiques*, (Vol. II), are not given here. I will only remark that the eight planets form two groups, the outer and inner. The first group includes Jupiter, Saturn, Uranus and Neptune, The roots g_1 to g_4 and γ_1 to γ_4 depend almost entirely on these four planets, and only the angles G_1 to G_4 and Γ_1 to Γ_4 appear in the secular terms with important coefficients. The angles G_5 to G_8 and Γ_5 to Γ_8 are important for the inner planets only, which, however also partake of the secular terms of the outer planets. It would naturally be expected, because of the smallness of the inner planets, that their g's and γ's and the appropriate coefficients, would be decidedly smaller than others. This is, however, not the case. For instance, Jupiter's mass is about 400 times as great as that of Venus, but Venus is only one fourteenth Jupiter's distance from the Earth; and as the attraction is as the inverse square of the distance, the ratio of the disturbing effect of Jupiter and Venus on the Earth is about 2:1. It is also to be added, that Venus's orbit has a decidedly larger inclination than Jupiter's, and, therefore, the perturbations of the ecliptic are largely due to Venus.

As the secular changes are very slow they may for centuries be taken as proportional to the time. To be entirely safe a term should be added which is proportional to the square of the time. Beginning with 1800, the eccentricity of the earth's orbit can be expressed by the formula

$$e = 0.01679207 - 0.0000004135t - 0.0000000000123t^2,$$

from which it appears that the eccentricity is decreasing, and at a growing rate.

The sixth table gives the course of the Earth's elements for a period of 200,000 years.

Elements of the Orbits of the Major Planets.

I.

Planet.	Mean Distance.	Eccentricity.	Epoch. Mean Moon. P = Paris. G = Greenwich.	Mean Longitude	Mean Daily (Sidereal) Motion.
Mercury......	0.3870988	0.2056048	Jan. 1, 1850. *P*.	327° 15′ 20″.4	14732″.41967
Venus..........	0.7233322	0.0068431	Jan. 1, 1850. *P*.	245 33 14.7	5767.66982
Earth	1.0000000	0.0167708	Jan. 1, 1850. *P*.	100 46 43.5	3548.19286
Mars...........	1.5236914	0.0932611	Jan. 1, 1850. *P*.	83 40 31.3	1886.51831
Jupiter........	5.20280	0.0482520	Jan. 1, 1850. *P*.	160 1 10.3	299.12836
Saturn	9.53886	0.0560717	Jan. 1, 1850. *P*.	14 52 28.3	120.45465
Uranus........	19.18336	0.0463592	Jan 0, 1850. *G*.	29 12 43.7	42.23079
Neptune......	30.05674	0.0084962	Jan. 0, 1850. *G*.	335 5 38.9	21.53302

II.

Planet.	Longitude of Perihelion.	Longitude of Ascending Node.	Inclination.	Periodic Time in Sideral Days.	Periodic Time in Tropical Days.
Mercury.........	75° 7′ 13″.9	46° 33′ 8″.8	7° 0′ 7″.7	87.96926	87.96843
Venus............	129 27 14.5	75 19 52.3	3 23 34.8	224.70079	224.69544
Earth.............	100 21 21.5	.. ..	. .. .	365.25636	365.24220
Mars..............	333 17 53.7	48 23 53.1	1 51 2.3	686.97979	686.92972
Jupiter...........	11 54 58.4	98 56 17.0	1 18 41.4	4332.59	4330.60
Saturn...........	90 6 56.7	112 20 53.0	2 29 39.8	10759.23	10746.95
Uranus...........	170 38 48.7	73 14 37.6	0 46 20.9	30688.51	30588.90
Neptune..........	43 17 30.3	130 7 31.9	1 47 1.6	60186.64	59804.81

III.

Planet.	Distance from the Sun in millions of kilometers.		Mass in terms of the		Specific Gravity
	Greatest.	Least.	Sun.	Earth.	
Mercury......	69.37	45.71	$\frac{1}{7636440}$	0.04	4.50
Venus.........	108.25	106.78	$\frac{1}{412150}$	0.78	4.52
Earth..........	151.13	146.14	$\frac{1}{322800}$	1.00	5.56
Mars...........	247.60	205.36	$\frac{1}{3093500}$	0.10	3.98
Jupiter........	810.64	736.01	$\frac{1}{1047.6}$	308	1.37
Saturn........	1497.32	1338.32	$\frac{1}{3490}$	92	0.65
Uranus........	2983.53	2719.16	$\frac{1}{22600}$	14	1.13
Neptune......	4505.48	4429.57	$\frac{1}{19380}$	17	1.71
Sun............			1	322800	1.42

IV.

Planet.	Annual Variation of the				Limit of eccentricity.	Limit of Inclination to ecliptic of 1800.
	Eccentricity in units of 7th decimal place.	Longitude of		Inclination.		
		Perihelion.	Ascending node.			
Mercury..	+ 2.034	+55″.914	+42″.643	+0″.063	0.266	9°17′
Venus......	— 5.397	+49.462	+32.890	+0.045	0.087	5 19
Earth	— 4.244	+61.700	....	....	0.078	4 52
Mars......	+ 9.541	+66.242	+27.992	—0.024	0.142	7 09
Jupiter....	+16.678	+57.910	+36.382	—0.205	0.062	2 01
Saturn	—34.276	+70.403	+31.407	—0.140	0.085	2 33
Uranus....	+27.387	+50.256	+18.568	+0.025	0.065	2 34
Neptune...	+ 0.557	+51.014	+39.615	—0.346	0.017	2 21

V.

$g_1 = 0''.692870$	$\gamma_1 = 0$
$g_2 = 2.842232$	$\gamma_2 = -0.756015$
$g_3 = 3.780294$	$\gamma_3 = -3.106931$
$g_4 = 22.500087$	$\gamma_4 = -25.952538$
$g_5 = 5.2989$	$\gamma_5 = -4.795350$
$g_6 = 7.5747$	$\gamma_6 = -7.067951$
$g_7 = 17.1527$	$\gamma_7 = -17.468102$
$g_8 = 17.8633$	$\gamma_8 = -18.567871$

ELEMENTS OF THE ORBIT OF THE EARTH REFERRED TO ECLIPTIC OF 1800.

VI.

Epoch.		Eccentricity.	Inclination.	Longitude of Perihelion.	Longitude of Ascending node.
B. C.	98200	0.0473	3° 45′ 31″	316° 18′	96° 34′
	88200	0.0452	2 42 19	340 2	76 17
	78200	0.0398	1 18 58	4 13	73 47
	68200	0.0316	1 13 58	27 22	136 8
	58200	0.0218	2 36 42	46 8	136 9
	48200	0.0131	3 40 11	50 44	116 9
	38200	0.0109	4 3 1	28 36	91 59
	28200	0.0151	3 41 51	25 50	66 49
	18200	0.0188	2 44 12	44 00	41 34
B. C.	8200	0.0195	1 24 35	69 47	16 39
A. D.	1800	0.0168	0 0 0	99 30	. . . . .
	11800	0.0115	1 14 26	134 14	148 15
	21800	0.0047	2 7 46	192 22	124 29
	31800	0.0059	2 33 19	318 47	100 29
	41800	0.0124	2 27 53	6 25	75 31
	51800	0.0173	1 51 54	38 3	48 13
	61800	0.0199	1 51 52	64 31	10 47
	71800	0.0211	1 34 35	86 14	220 38
	81800	0.0188	1 45 40	101 38	170 15
	91800	0.0176	2 40 56	109 19	139 3
	101800	0.0189	3 2 57	114 5	109 57